INTERTWINED SYNCHRONIZED SYSTEMS

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Abstract. An asymmetric-RLL(d_1, k_1, d_0, k_0) system is a subshift of $\{0, 1\}^{\mathbb{Z}}$ with run of 1 and 0 restricted to $S = [d_1, k_1] \subseteq \mathbb{N}_0 = \mathbb{N} \cup \{0\}$ and $S' = [d_0, k_0] \subseteq \mathbb{N}_0$ respectively. We extend this concept to the case when S and S' are arbitrary subsets of $\mathbb{N}_0$ and we call it a (S, S')-gap shift. Moreover, for $i = 1, 2$, if X_i is a synchronized system generated by $V_i = \{v^i\alpha_i : \alpha_i v^i \alpha_i \in \mathcal{B}(X_i), \alpha_i \not\subseteq v^i\}$ where α_i is a synchronizing word for X_i, then a natural generalization of (S, S')-gap shifts is a coded system Z generated by $\{v^1\alpha_1 v^2\alpha_2 : v^i\alpha_i \in V_i, i = 1, 2\}$ and called the intertwined system. We investigate the dynamical properties of Z with respect to X_1 and X_2.

Introduction

Recall that the Run-length-limited (RLL) [11] and the Maximum Transition Run (MTR) constrained systems [13] are used to improve timing and detection performance in storage channels. In particular, the MTR code, introduced by Moon and Brickner [13], are to provide coding gain for extended partial response channels. The RLL code denoted by $X(d, k)$ limits the run of 0 to be at least d and at most k whereas the MTR(j, k) code limits the run of 0 to be at most k and the run of 1 at most j. When there is no restriction on the runs of 0, we say that $k = \infty$ and it is common then to denote such a constraint by MTR(j). For generalizing MTR codes, consider the asymmetric-RLL(d_1, k_1, d_0, k_0) constraint which is the set of binary sequences whose runs of 1's have length between d_1 and k_1 and the runs of 0's between d_0 and k_0. In the case that $d_1 = d_0 = 1$, $k_1 = j$ and $k_0 = k$, this constraint coincides with MTR(j, k). The S-gap shifts in abstract symbolic dynamical system, may be considered as a generalization of the RLL codes when the run of 0 is restricted to a subset $S \subseteq \mathbb{N} \cup \{0\}$. In this note, we extend these concepts and introduce the (S, S')-gap shifts as a generalization of S-gap shifts on one side and also a generalization of the asymmetric-RLL(d_1, k_1, d_0, k_0) constrained systems when the run of 0 and the run of 1 are restricted to subsets of $\mathbb{N}_0$, S and S' respectively. To be more specific, fix two increasing sets S and S' in $\mathbb{N}_0$. Also let $V = \{v_s = 0^s1 : s \in S\}$ and $W = \{v_{s'} = 1^{s'}0 : s' \in S'\}$. Then V and W generate two S-gap shifts $X(S)$ and $X(S')$ and the coded system generated by $U = \{v_s v_{s'} : s \in S, s' \in S'\}$ is called a (S, S')-gap shift and is denoted by $X(S, S')$. When $S' = \{0\}$, then $X(S, S')$ will be $X(S + 1)$. We will investigate some similarities and differences in dynamical properties of the S-gap shifts with (S, S')-gap shifts and with respect to S and S'.

Note that 1 (resp. 0) is a synchronizing word for $X(S)$ (resp. $X(S')$). Also recall that a synchronized system X with a synchronizing word α is generated by

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$\{v\alpha : \alpha v\alpha$ is a word in X and $\alpha \nsubseteq v\}$. If Y is another synchronized system with a synchronized word β and a set of generators $W_\beta = \{w\beta : \beta w\beta$ a word in Y and $\beta \nsubseteq w\}$, then a generalization for a $(S,\, S')$-gap shift is a coded system generated by $\{v\alpha w\beta : v\alpha \in V_\alpha,\, w\beta \in W_\beta\}$. We call this new system an *intertwined system* of X and Y. Dynamical properties of an intertwined system depend on α and β; however, here, we are interested in those dynamical properties which are independent of the synchronized words. In fact, first we will study the intertwined systems and then will look for further properties of $(S,\, S')$-gap shifts that may not be held or considered by intertwined systems. In particular, we will give a formula for the entropy of a $(S,\, S')$-gap shift and will compute the Bowen-Franks groups when it is SFT.

1. Background and Notations

This section is devoted to the very basic definitions in symbolic dynamics. The notations has been taken from [11] and the proofs of the relevant claims in this section can be found there. Let $\mathcal{A}$ be an alphabet, that is a non-empty finite set. The full $\mathcal{A}$-shift denoted by $\mathcal{A}^{\mathbb{Z}}$, is the collection of all bi-infinite sequences of symbols in $\mathcal{A}$. A block (or word) over $\mathcal{A}$ is a finite sequence of symbols from $\mathcal{A}$. It is convenient to include the sequence of no symbols, called the empty word and denoted by ε. If x is a point in $\mathcal{A}^{\mathbb{Z}}$ and $i \leq j$, then we will denote a word of length $j-i$ by $x_{[i,\,j]} = x_i x_{i+1} \ldots x_j$. If $n \geq 1$, then u^n denotes the concatenation of n copies of u, and put $u^0 = \varepsilon$. The shift map σ on the full shift $\mathcal{A}^{\mathbb{Z}}$ maps a point x to the point $y = \sigma(x)$ whose ith coordinate is $y_i = x_{i+1}$.

Let $\mathcal{F}$ be the collection of all forbidden blocks over $\mathcal{A}$. The complement of $\mathcal{F}$ is the set of *admissible blocks* or just *words* in X. For any such $\mathcal{A}^{\mathbb{Z}}$, define $X_{\mathcal{F}}$ to be the subset of sequences in $\mathcal{A}^{\mathbb{Z}}$ not containing any word in $\mathcal{F}$. A shift space is a closed subset X of a full shift $\mathcal{A}^{\mathbb{Z}}$ such that $X = X_{\mathcal{F}}$ for some collection $\mathcal{F}$ of forbidden words over $\mathcal{A}$.

Let $\mathcal{B}_n(X)$ denote the set of all admissible n blocks. The *Language* of X is the collection $\mathcal{B}(X) = \bigcup_{n=0}^{\infty} \mathcal{B}_n(X)$. A shift space X is *irreducible* if for every ordered pair of blocks $u, v \in \mathcal{B}(X)$ there is a word $w \in \mathcal{B}(X)$ so that $uwv \in \mathcal{B}(X)$ and it is *totally transitive* if σ^n is irreducible for all $n \in \mathbb{N}$ where σ_S is the shift map defined on X. It is called *weak mixing* if for every ordered pair $u, v \in \mathcal{B}(X)$, there is a thick set (a subset of integers containing arbitrarily long blocks of consecutive integers) P such that for every $n \in P$, there is a word $w \in \mathcal{B}_n(X)$ such that $uwv \in \mathcal{B}(X)$. It is *mixing* if for every ordered pair $u, v \in \mathcal{B}(X)$, there is an N such that for each $n \geq N$ there is a word $w \in \mathcal{B}_n(X)$ such that $uwv \in \mathcal{B}(X)$. A word $v \in \mathcal{B}(X)$ is *synchronizing* if whenever uv and vw are in $\mathcal{B}(X)$, we have $uvw \in \mathcal{B}(X)$. An irreducible shift space X is a *synchronized system* if it has a synchronizing word [5].

Let $\mathcal{A}$ and $\mathcal{D}$ be alphabets and X a shift space over $\mathcal{A}$. Fix integers m and n with $m \leq n$. Define the $(m+n+1)$-*block map* $\Phi : \mathcal{B}_{m+n+1}(X) \to \mathcal{D}$ by

$$y_i = \Phi(x_{i-m} x_{i-m+1} \ldots x_{i+n}) = \Phi(x_{[i-m,i+n]}) \tag{1.1}$$

where y_i is a symbol in $\mathcal{D}$. The map $\phi : X \to \mathcal{D}^{\mathbb{Z}}$ defined by $y = \phi(x)$ with y_i given by (2.1) is called the *sliding block code* with *memory* m and *anticipation* n induced by Φ. An onto sliding block code $\phi : X \to Y$ is called a factor code. In this case, we say that Y is a factor of X. The map ϕ is a *conjugacy*, if it is invertible.

A shift space X is called a *shift of finite type* (SFT) if there is a finite set $\mathcal{F}$ of forbidden words such that $X = X_{\mathcal{F}}$. An *edge shift*, denoted by X_G, is a shift space which consist of all bi-infinite walks in a directed graph G. Each edge e initiates at a vertex denoted by $i(e)$ and terminates at a vertex $t(e)$.

A *labeled graph* $\mathcal{G}$ is a pair $(G, \mathcal{L})$ where G is a graph with edge set $\mathcal{E}$, and the labeling $\mathcal{L} : \mathcal{E} \to \mathcal{A}$. A *sofic shift* $X_{\mathcal{G}}$ is the set of sequences obtained by reading the labels of walks on G,

$$X_{\mathcal{G}} = \{\mathcal{L}_\infty(\xi) : \xi \in X_G\} = \mathcal{L}_\infty(X_G). \tag{1.2}$$

We say $\mathcal{G}$ is a *presentation* of $X_{\mathcal{G}}$. Every SFT is sofic [11, Theorem 3.1.5], but the converse is not true.

A labeled graph $\mathcal{G} = (G, \mathcal{L})$ is *right-resolving* if for each vertex I of G the edges starting at I carry different labels. A *minimal right-resolving presentation* of a sofic shift X is a right-resolving presentation of X having the fewer vertices among all right-resolving presentations of X. Any two minimal right-resolving presentations of an irreducible sofic shift must be isomorphic as labeled graphs [11, Theorem 3.3.18]. So we can speak of the minimal right-resolving presentation of an irreducible sofic shift X, which we call it the *Fischer cover* of X.

Let X be a shift space and $w \in \mathcal{B}(X)$. The *follower set* $F(w)$ of w is defined by $F(w) = \{v \in \mathcal{B}(X) : wv \in \mathcal{B}(X)\}$. A shift space X is sofic if and only if it has a finite number of follower sets [11, Theorem 3.2.10]. In this case, we have a labeled graph $\mathcal{G} = (G, \mathcal{L})$ called the *follower set graph* of X. The vertices of G are the follower sets and if $wa \in \mathcal{B}(X)$, then draw an edge labeled a from $F(w)$ to $F(wa)$. If $wa \notin \mathcal{B}(X)$ then do nothing.

A labeled graph is *right-closing* with delay D if whenever two paths of length $D + 1$ start at the same vertex and have the same label, then they must have the same initial edge. Similarly, left-closing will be defined. A labeled graph is bi-closing, if it is simultaneously right-closing and left-closing.

An irreducible sofic shift is called *almost-finite-type* (AFT) if it has a bi-closing presentation [11]. Since any sofic shift is a factor of an SFT, it is clear that an AFT is sofic. Nasu [15] showed that an irreducible sofic shift is AFT if and only if its minimal right-resolving presentation is left-closing.

Set $\mathcal{F}$ to be a finite collection of words over a finite alphabet $\mathcal{A}$ where each $w_j \in \mathcal{F}$ is associated with a non-negative integer index n_j. Write

$$\mathcal{F} = \{w_1^{(n_1)}, w_2^{(n_2)}, ..., w_{|\mathcal{F}|}^{(n_{|\mathcal{F}|})}\} \tag{1.3}$$

and associate with the indexed list $\mathcal{F}$ a period T, where T is a positive integer satisfying $T \geq \max\{n_1, n_2, ..., n_{|\mathcal{F}|}\} + 1$.

A shift space X is a shift of *periodic-finite-type* (PFT) if there exists a pair $\{\mathcal{F}, T\}$ with $|\mathcal{F}|$ and T finite so that $X = X_{\{\mathcal{F},T\}}$ is the set of bi-infinite sequences that can be shifted such that the shifted sequence does not contain a word $w_j^{n_j} \in \mathcal{F}$ starting at any index m with $m \mod T = n_j$. A strictly PFT shift cannot be represented as an SFT.

Let $\mathcal{G}$ be the minimal right-resolving presentation of an irreducible sofic shift, $p = \mathrm{per}(A_{\mathcal{G}})$ and $D_0, D_1, ..., D_{p-1}$ the period classes of $\mathcal{G}$. An indexed word $w^{(n)} = (w_0, w_1, ..., w_{l-1})^{(n)}$ is a *periodic first offender of period class* n if $w \notin \cup_{I \in D_n} F_{\mathcal{G}}(I)$ but for all $i, j \in [0, l-1]$ with $i \leq j$ and $w_{[i,j]} \neq w$, $w_{[i,j]} \in \cup_{I \in D_{(n+i) \mod p}} F_{\mathcal{G}}(I)$.

An irreducible sofic shift is PFT if and only if the set of periodic first offenders is finite [4, Corollary 14].

Now we review the concept of Fischer cover for a not necessarily sofic system [8]. Let $x \in \mathcal{B}(X)$. Then $x_+ = (x_i)_{i\in\mathbb{Z}^+}$ (resp. $x_- = (x_i)_{i<0}$) is called *right (resp. left) infinite X-ray*. For a left infinite X-ray, say x_-, its follower set is $\omega_+(x_-) = \{x_+ \in X^+ : x_-x_+$ is a point in $X\}$. Consider the collection of all follower sets $\omega_+(x_-)$ as the set of vertices of a graph X^+. There is an edge from I_1 to I_2 labeled a if and only if there is an X-ray x_- such that x_-a is an X-ray and $I_1 = \omega_+(x_-)$, $I_2 = \omega_+(x_-a)$. This labeled graph is called the *Krieger graph* for X. If X is a synchronized system with synchronizing word α, the irreducible component of the Krieger graph containing the vertex $\omega_+(\alpha)$ is called the *right Fischer cover* of X.

2. Coded Systems

A shift space that can be presented by an irreducible countable labeled graph is called a *coded system*. Equivalently, a coded system X is the closure of the set of sequences obtained by freely concatenating the words in a list of words, called the set of generators, over a finite alphabet [11]. A coded system is irreducible and has a dense set of periodic points [8]. Coded systems were introduced by Blanchard and Hansel [5] who also showed that the class of the coded systems is the smallest class of subshifts which contains the synchronized systems and is closed under factors [5, proposition 4.1]. A brief introduction to coded systems can be found in [11, section 13.5].

In a synchronized system X, for any synchronized word $\alpha = \alpha_1 \cdots \alpha_p$, X is generated by

$$V = V_\alpha = \{v\alpha \in \mathcal{B}(X) : \alpha v\alpha \in \mathcal{B}(X),\, \alpha \not\subseteq v\}. \tag{2.1}$$

Let $\mathcal{G} = (G, \mathcal{L})$ be the right Fischer cover of X. We give another right resolving cover for X, denoted by $\mathcal{G}^\alpha$ which is not necessarily follower-separated. Suppose $\mathbf{v}(\alpha)$ is the unique vertex in $\mathcal{V}(G)$ where any path labeled α terminates at. Any other vertex v is shown by $\mathbf{v}(\alpha u)$ where u is some path starting at $\mathbf{v}(\alpha)$ and terminating at v. To be specific, we adopt the convention that if there is u' such that $\mathbf{v}(\alpha u') = \mathbf{v}(\alpha u)$, then $|u| \le |u'|$. Fix $I = \mathbf{v}(\alpha u_1) \in \mathcal{V}(G)$ and assume that $\mathcal{E}_I$, the set of inner edges of I, has more than one element. This implies that by tracking the paths starting at $\mathbf{v}(\alpha)$, there is another path u_2, $|u_2| \ge |u_1|$ such that $I = \mathbf{v}(\alpha u_1) = \mathbf{v}(\alpha u_2)$. Suppose $u_i = c_{i_1} \cdots c_{i_{k_i}} \alpha_1 \cdots \alpha_{l_i} \in \mathcal{B}(X)$, $i = 1$ or 2. We intend to do some in-split [11, §2.4] for v. If one of the following holds, then we do not do the splitting.

1. both $\alpha_1 \cdots \alpha_{l_1}$ and $\alpha_1 \cdots \alpha_{l_2}$ are empty words;
2. $\alpha_1 \cdots \alpha_{l_1}$ (resp. $\alpha_1 \cdots \alpha_{l_2}$) is not empty word and $c_{i_1} \cdots c_{i_{k_1}} \alpha_1 \cdots \alpha_{l_1} \cdots \alpha_p = c_{i_1} \cdots c_{i_{k_1}} \alpha$ (resp. $c_{i_1} \cdots c_{i_{k_2}} \alpha$) is not admissible;
3. cases (1) and (2) do not hold and $l_1 = l_2$.

(1) and (2) say that if v is not a vertex on a path π_α labeled α, then in-splitting will not be done.

Now we set up to see which vertices on π_α requires in-splitting and how this happens. Note that case (3) above excludes some cases. Set $\mathcal{G}_1 = \mathcal{G}$ and let $\mathcal{V}_{G_1}(\alpha_1) = \{I \in \mathcal{V}(G_1) : I = t(e_{\alpha_1}),\, e_{\alpha_1}$ is the first edge labeled α_1 on a path labeled $\alpha\}$. For $I \in \mathcal{V}_{G_1}(\alpha_1)$, partition $\mathcal{E}_I$ to $P_I^1(\alpha_1) = \{e_1 : \mathcal{L}(e_1) = \alpha_1\}$ and

$P_I^2(\alpha_1)$ for the remaining edges. Do an in-split for I with respect to this partition and call the new cover $\mathcal{G}_2 = (G_2, \mathcal{L}_2)$.

Let $\mathcal{V}_{G_2}(\alpha_1\alpha_2) = \{I \in \mathcal{V}(G_2) : I = t(e_{\alpha_1}e_{\alpha_2}), e_{\alpha_1}e_{\alpha_2}$ be the first 2 edges with label $\alpha_1\alpha_2$ of a path labeled $\alpha\}$. Partition $\mathcal{E}_I$, $I \in \mathcal{V}_{G_2}(\alpha_1\alpha_2)$ to $P_I^1(\alpha_1\alpha_2) = \{e_2 : t(e_1e_2) = I$ for some e_1, $\mathcal{L}_2(e_1e_2) = \alpha_1\alpha_2\}$ and $P_I^2(\alpha_1\alpha_2) = \mathcal{E}_I \setminus P_I^1(\alpha_1\alpha_2)$. By the same procedure, $\mathcal{G}_{k+1}$, $P_I^1(\alpha_1\alpha_2\cdots\alpha_k)$, and $P_I^2(\alpha_1\alpha_2\cdots\alpha_k)$, $1 \le k < p$ will be constructed. Set $\mathcal{G}^\alpha = \mathcal{G}_p = (G_p, \mathcal{L}_{p-1})$. Suppose in-splitting occurs at $I \in \mathcal{V}(G_k)$ and let $\mathcal{E}^I$ be the set of outer edges of I. Then corresponding to I, there are two vertices I_1 and I_2 in $\mathcal{V}(G_{k+1})$ with $\mathcal{E}^I = \mathcal{E}^{I_1} = \mathcal{E}^{I_2}$. For $e \in \mathcal{E}$ let $e(i)$ be the corresponding edge in $\mathcal{E}^{I_i}$ with the same label as e. We collect some properties of $\mathcal{G}^\alpha$ in the following theorem.

Theorem 2.1. *Let X be a synchronized system with a synchronized word $\alpha = \alpha_1\cdots\alpha_p$, a generator V as (2.1) and the right Fischer cover $\mathcal{G} = (G, \mathcal{L})$. Then*

1. *$\mathcal{G}^\alpha$ and $\mathcal{G}$ are conjugate.*
2. *Let $e_{\alpha_1}\cdots e_{\alpha_k}$, $1 \le k < p$ be a subpath of a path labeled α, $\mathcal{L}_p(e_{\alpha_1}\cdots e_{\alpha_k}) = \alpha_1\cdots\alpha_k$ and let $I = t(e_{\alpha_1}\cdots e_{\alpha_k})$. Then all the inner edges of I have the same label α_k.*
3. *Let $u = u_1\cdots u_k \in \mathcal{B}(X)$ and suppose $\pi = e_{u_1}\cdots e_{u_k}e_{\alpha_i}\cdots e_{\alpha_p}$ is a path so that $\mathcal{L}_p(\pi) = u_1\cdots u_k\alpha_i\cdots\alpha_p$ and $e_{\alpha_i}\cdots e_{\alpha_p}$ is a subpath of a path labeled α, then either $u\alpha_i\cdots\alpha_p \subseteq \alpha$ or $u\alpha_i\cdots\alpha_p = v\alpha_1\cdots\alpha_p$ for some $v \in \mathcal{B}(X)$.*
4. *If X is sofic, then $\mathcal{G}^\alpha$ is a finite labeled graph. Also, if $\mathcal{G}$ is left-closing with delay D, then $\mathcal{G}^\alpha$ will be left-closing with delay $D + p - 1$.*

Proof. (2), (3) and the first part of (4) follow from the definition of $\mathcal{G}^\alpha$. The second part of (4) is satisfied by [11, Proposition 5.1.8]. (1) can be deduced from [11, Theorem 2.4.10]; nevertheless, we give a sketch of proof. Set $X_G = X_{G_1}$. First we show that X_{G_1} and X_{G_2} are conjugate; then an induction argument gives the result. Define a 1-block map $\Psi : \mathcal{B}_1(X_{G_2}) \to \mathcal{B}_1(X_{G_1})$ by $\Psi(e(j)) = e$. Note that the image under Ψ of any path in G_2 is a path in G_1. Next we define a 2-block map $\Phi : \mathcal{B}_2(X_{G_1}) \to \mathcal{B}_1(X_{G_2})$. If $fe \in \mathcal{B}_2(X_{G_1})$, then f is in $P_I^i(\alpha_1)$, $i \in \{1, 2\}$ and we define $\Phi(fe) = e(i)$. Here also, the image under Φ of any path in G_1 is a path in G_2. Let $\psi = \Psi_\infty$ and $\phi = \Phi_\infty^{[0,1]}$. Then $\psi \circ \phi$ and $\phi \circ \psi$ are identity maps and the conjugacy will be achieved. □

Theorem 2.2. [18] *A coded system Z with the set of generators $\{z_n\}$ is mixing if and only if* $\gcd\{|z_n| : n \in \mathbb{N}\} = 1$.

Theorem 2.3. *Let X be a synchronized system. Then the following is equivalent.*

1. *X is mixing.*
2. *X is weak mixing.*
3. *X is totally irreducible.*

Proof. Clearly every mixing shift space is weak mixing. Furthermore, any weak mixing shift space is totally irreducible [7].

(3) $\Rightarrow$ (2) Because if X is totally irreducible and has a dense set of periodic points, it is weak mixing [3]. It remains to show that a weak mixing X is mixing. Let α is a synchronizing word for X. So X is generated by $\{v_n\alpha : n \in \mathbb{N}\}$. By Lemma 2.2, it suffices to show $\gcd\{|v_n| + |\alpha| : n \in \mathbb{N}\} = 1$. Since X is weak mixing, there is a thick set P such that for every $n \in P$ there exists a word $w \in \mathcal{B}_n(X)$

with $\alpha w\alpha \in \mathcal{B}(X)$. Thus there are words of length $m, m+1 \in P$ of the form αu, implying that $\gcd\{|v_n| + |\alpha| : n \in \mathbb{N}\} = 1$. □

3. Intertwined Synchronized Systems

Definition 3.1. For $1 \leq i \leq \ell$, let $X_i = X_{V_i}$ be a coded system with a synchronizing word α_i and generated by

$$V_i = V_{\alpha_i} = \{v^i\alpha_i : \alpha_i v^i \alpha_i \in \mathcal{B}(X_i),\ \alpha_i \not\subseteq v^i\}.$$

The coded system $Z = Z(V_1, \dots, V_\ell)$ generated by

$$\{v^1\alpha_1 v^2\alpha_2 \cdots v^\ell \alpha_\ell :\ v^i\alpha_i \in V_i\}$$

is called the *intertwined system* of $X_1, \dots, X_\ell$ and is denoted by

$$Z = X_1 \& X_2 \& \cdots \& X_\ell.$$

Let $\mathcal{G}_i = (G_i, \mathcal{L}_i)$ be the right Fischer cover for X_i and $\mathcal{G}_i^{\alpha_i}$ the cover provided by Theorem 2.1 for X_i, $1 \leq i \leq \ell$. Then all α_i-paths, that is paths labeled α_i in $G_i^{\alpha_i}$ terminate at $\mathbf{v}(\alpha_i)$. Now $\mathcal{G}_Z = \mathcal{G}_Z(\mathcal{G}_{X_1}, \dots, \mathcal{G}_{X_\ell})$ which is a cover obtained by choosing α_i-paths being terminated at $\mathbf{v}(\alpha_{(i+1) \mod \ell})$ instead of $\mathbf{v}(\alpha_i)$ is a cover for Z. We call $\mathcal{G}_Z$ the *intertwined cover* for Z. So to construct $\mathcal{G}_Z$, we cut off all the α_i-paths from $\mathbf{v}(\alpha_i)$ in $G_i^{\alpha_i}$ and will glue them to $\mathbf{v}(\alpha_j)$ in $G_j^{\alpha_j}$ where $j = i+1 \mod \ell$.

Since the problems arising from intertwinning of finite systems are basically the same as intertwinning of two systems, from now on we concentrate on intertwinning of two systems $X = X_V$ and $Y = Y_W$ generated by

(3.1)
$$V = V_\alpha = \{v\alpha :\ \alpha v\alpha \in \mathcal{B}(X), \alpha \not\subseteq v\} \quad \text{and} \quad W = W_\beta = \{w\beta :\ \beta w\beta \in \mathcal{B}(Y), \beta \not\subseteq w\}$$

respectively. So α-paths (resp. β-paths) terminate at $\mathbf{v}(\beta)$ (resp. $\mathbf{v}(\alpha)$).

Definition 3.2. Let G_{Z_X} be the subgraph of G_Z corresponding to G_X, that is, consisting of all the paths in G_Z labeled $v\alpha$, $v \in V$ and starting from $\mathbf{v}(\alpha)$ and terminating at $\mathbf{v}(\beta)$.

Note that $\mathcal{G}_{Z_X}$ is not irreducible and $\mathcal{G}_{Z_X}$ and $\mathcal{G}_{Z_Y}$ have only vertices $\mathbf{v}(\alpha)$ and $\mathbf{v}(\beta)$ in common.

Now by giving an example, we show that $X_V \& X_W$ depends on V and W and so on α and β. Let $X = X(S)$ be a S-gap shift [11] for $S = \{1, 2\}$ and Y a β-shift [20] for $1_\beta = 1101$; both systems are SFT. It is obvious that 1 is a synchronizing word for X and 00, 11 are synchronizing words for Y. Let $V = \{01, 001\}$, $W_1 = \{u00 : 00u00 \in \mathcal{B}(Y),\ 00 \not\subseteq u\}$ and $W_2 = \{u11 : 11u11 \in \mathcal{B}(Y),\ 11 \not\subseteq u\}$. Then $X = X_V$ and $Y = X_{W_1} = X_{W_2}$. By a straightforward routine, that is by finding the largest positive eigenvalues of adjacency matrices, one has $h(X \& X_{W_1}) = \log 1.63$ while $h(X \& X_{W_2}) = \log 1.7$. So $X \& X_{W_1}$ and $X \& X_{W_2}$ are not conjugate. In particular, the intertwined system of conjugate systems are not necessarily conjugate.

Theorem 3.3. *Let X and Y be two synchronized systems generated by $V = V_\alpha$ and $W = W_\beta$ as in 3.1. Then X and Y are sofic if and only if $Z = X \& Y = X_V \& Y_W$ is sofic.*

Proof. Let $X = X_V$ and $Y = X_W$ be two sofic systems and $\mathcal{G}_X$ (resp. $\mathcal{G}_Y$) be the Fischer covers of X (resp. Y). Then $\mathcal{G}_X^\alpha$ and $\mathcal{G}_Y^\beta$ and their intertwined cover $\mathcal{G}_Z$

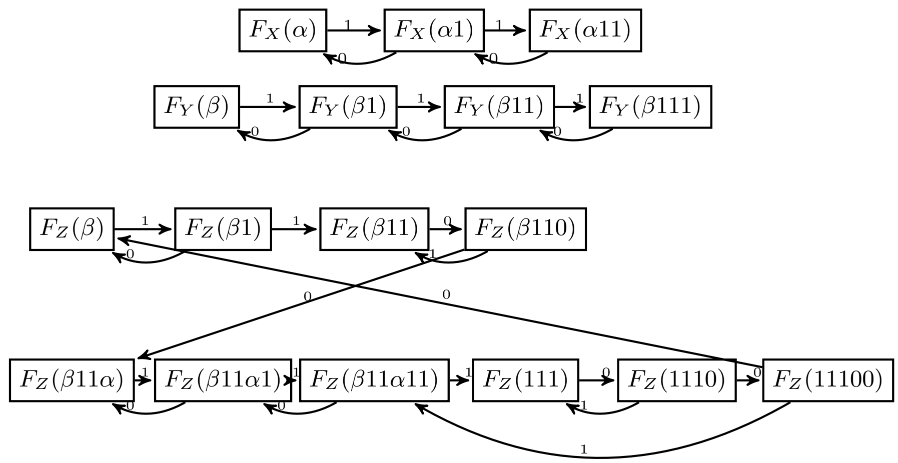

FIGURE 1. From above, the Fischer covers of X, Y and $Z = X\&Y$.

have finite vertices. But any symbolic system with a finite labeled graph is sofic and we are done.

For the converse if Z is sofic but X is not sofic, then G_{Z_X} will be a graph with infinite vertices which is absurd. $\square$

Next example will illustrate the intertwinning of two sofic systems X and Y.

Example 3.4. Consider Figure 1 and two sofic shifts X and Y with $\alpha = \alpha_1\alpha_2 = 00$ and $\beta = \beta_1\beta_2\beta_3 = 000$ as their synchronizing words respectively. The Fischer covers of X and Y are presented in that figure.

First we will construct $\mathcal{G}_X^\alpha$. We have $\mathcal{V}_{G_X}(\alpha_1) = \{F_X(\alpha_1)\}$ and only $I = F_X(\alpha_1)$ needs in-splitting. We do this and we obtain $\mathcal{G}_X^\alpha = \mathcal{G}_{X_2}$.

For $\mathcal{G}_Y^\beta$, the first in-splitting occurs in $I = F_Y(\beta 11)\}$. Do this in-splitting and call the new cover $\mathcal{G}_{Y_2}$. We have $\mathcal{V}_{G_{Y_2}}(\beta_1\beta_2) = \{F_Y(\beta 1)\}$ and $F_Y(\beta 1)$ needs also in-splitting. Doing this $\mathcal{G}_Y^\beta (= \mathcal{G}_{Y_3})$ will be constructed.

Definition 3.5. [10] A shift space X has *specification with variable gap length* (SVGL) if there exists $N \in \mathbb{N}$ such that for all $u, v \in \mathcal{B}(X)$, there exists $w \in \mathcal{B}(X)$ with $uwv \in \mathcal{B}(X)$ and $|w| \leq N$.

Note that a SVGL was called almost specified in [10].

Theorem 3.6. *Suppose X and Y are two synchronized systems generated by $V = V_\alpha$ and $W = W_\beta$ as in 3.1. Then $Z = X\&Y = X_V\&Y_W$ has SVGL if and only if $X = X_V$ and $Y = Y_W$ have SVGL.*

Proof. If $V = W$, then $Z = X$ and we are done. So suppose $W \neq V$ and pick $w_0\beta \in W \setminus V$.

First suppose Z has SVGL with the transition length M and suppose that one of X or Y, say X, does not have SVGL. Then for all n, there are $u_n, v_n \in \mathcal{B}(X)$

such that if $w \in \mathcal{B}(X)$ and $u_n w v_n \mathcal{B}(X)$, then $|w| \geq n$. Without loss of generality, assume that $\alpha u_n, v_n\alpha \in \mathcal{B}(X)$ for all n. Now let $z_n = \alpha w_0 \beta u_n$ and $z'_n = v_n \alpha w_0 \beta$ be the words in $\mathcal{B}(Z)$. Since Z has SVGL, there is $z''_n \in \mathcal{B}(Z)$ such that $z_n z''_n z'_n \in \mathcal{B}(Z)$ and $|z''_n| \leq M$ for all $n \in \mathbb{N}$. Note that this z''_n is a word such as $u'_n \alpha w_{i_1} \beta \cdots w_{i_k} \beta v'_n$ for some u'_n, $v'_n \in \mathcal{B}(X)$. Let $n > M$ and set $w = u'_n \alpha v'_n$. Then by the fact that α is a synchronizing word, $u_n w v_n \in X$ and $|w| \leq M$ which is absurd.

Now suppose both of X and Y have SVGL with the transition lengths M_X and M_Y. Let

$$\begin{aligned} m_1 &= \min\{|v\alpha| : v\alpha \in V\} = |v_1\alpha|, \qquad m_2 = \min\{|w\beta| : w\beta \in W\} = |w_1\beta|, \\ k &= \max\{n \in \mathbb{N} : n\alpha < M_X\}, \qquad\qquad l = \max\{n \in \mathbb{N} : n\beta < M_Y\}, \end{aligned}$$

and $M = M_X + km_2 + M_Y + lm_1$. We claim that M is a transition length for Z. Let $z_1, z_2 \in \mathcal{B}(Z)$. Different cases occur. We just prove two cases, other cases will be proved similarly. First case is when $z_1 = \gamma v_i \alpha w_j \beta z'$ and $z_2 = z'' \alpha w_p \beta \lambda$ where $\gamma, \lambda \in \mathcal{B}(Z)$ and z', $z'' \in \mathcal{B}(X)$ so that $\alpha \not\subseteq z'$, z''. Since X has SVGL, there is $x = x_1 \alpha v_{i_1} \alpha \cdots v_{i_n} \alpha x_2$ such that $z' x z'' \in \mathcal{B}(X)$ and $|x| \leq M_X$. Then $z = x_1 \alpha w_1 \beta v_{i_1} \alpha w_1 \beta \cdots v_{i_n} \alpha w_1 \beta x_2 \in \mathcal{B}(Z)$ and $z_1 z z_2 \in \mathcal{B}(Z)$. Furthermore, $|z| \leq M_X + km_2 \leq M$.

The other case is when z_1 is as above and $z_2 = z'' \beta v_q \alpha \lambda$ with $\beta \not\subseteq z'' \in \mathcal{B}(Y)$. Since X and Y have SVGL, there are $x \in \mathcal{B}(X)$ and $y \in \mathcal{B}(Y)$ such that $z' x \alpha \in \mathcal{B}(X)$, $\beta y z'' \in \mathcal{B}(Y)$ and $|x| \leq M_X, |y| \leq M_Y$. We can assume that x (resp. y) does not contain α (resp. β) as a subword. Then $z_1 x \alpha y z_2 \in \mathcal{B}(Z)$. Note that $|x\alpha y| \leq M_X + m_1 + M_Y \leq M$ and we are done. □

Recall that when X is a sofic shift space with non-wandering part $R(X)$, we can consider the shift space

$$\partial X = \{x \in R(X) : x \text{ contains no words that are synchronizing for } R(X)\}$$

which is called the *derived shift space* of X. An irreducible sofic shift space X is *near Markov* when it is AFT and its derived shift space ∂X is a finite set [19].

Theorem 3.7. *Let X and Y be two synchronized systems with $V = V_\alpha$ and $W = W_\beta$ generators for X and Y as in 3.1. If $Z = X \& Y = X_V \& Y_W$ is SFT, near Markov or AFT, then both X and Y are SFT, near Markov or AFT respectively.*

Proof. By definition of derived shift space, if $x \in \partial X$, α is not a subword of x and then $x \in \partial Z$. That is, $\partial X \subseteq \partial Z$. By the same reasoning, $\partial Y \subseteq \partial Z$ and so

$$\partial X \cup \partial Y \subseteq \partial Z. \tag{3.2}$$

First suppose Z is SFT and one of X or Y, say X, is not SFT. Then $\partial X \neq \emptyset$ however $\partial Z = \emptyset$. So X is SFT.

Now suppose X is not AFT. So there are two different infinite paths $x = \cdots e_{-1}e_0$ and $x' = \cdots e'_{-1}e'_0$ with the same label and $t(e_0) = t(e'_0)$. If $\alpha \not\subseteq \mathcal{L}_{X_\infty}(x) = \mathcal{L}_{X_\infty}(x')$, then x and x' will be two paths in G_Z where $\mathcal{L}_{Z_\infty}(x) = \mathcal{L}_{Z_\infty}(x')$ and terminating at the same vertex of $\mathcal{V}(G_Z)$. So Z is not AFT which is absurd. Otherwise, since α is a synchronizing word and so magic for $\mathcal{G}_X$, we may assume $\mathcal{L}_X(e_{-(|\alpha|-1)} \cdots e_{-1}e_0) = \alpha$ and by the proof of Theorem 3.3, both of these paths terminate at the same vertex. By technique of merging [11, §3.3], one can obtain the Fischer cover of Z from $\mathcal{G}_Z$. However, two vertices of $\mathcal{G}_Z$ merge only if one in $\mathcal{V}(G_{Z_X})$ and the other is in $\mathcal{V}(G_{Z_Y})$. Hence after merging, x and x' will be yet

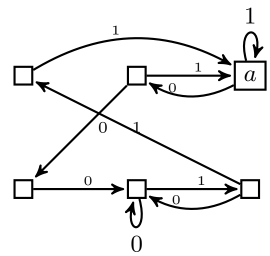

Figure 2. The Fischer cover of a non-AFT intertwined system, constructed from two AFT systems: two different paths labeled $(01)^\infty 111$ terminate at vertex a.

two different paths with the same label and terminating at the same vertex. This means Z is not AFT which is absurd.

If Z is near Markov, then it is AFT and $|\partial Z| < \infty$. So X and Y are near Markov if ∂X and ∂Y are finite which is consequence of 3.2. $\square$

The converse in Theorem 3.7 does not hold necessarily. We will give an example of X and Y, both AFT, in fact SFT, such that $X_V \& Y_W$ is not AFT for some set of generators V and W.

Example 3.8. Let $S = S' = \{0, 1, 2\}$, $X = X(S)$ and take Y to be the set of binary sequences whose runs of 1's is restricted to S'. Choose $\alpha = 00$ and $\beta = 11$ to be the synchronized words for defining the generating sets V and W respectively. The Fischer cover of $X \& Y = X_V \& Y_W$ is as in Figure 2. Observe that there are two different infinite paths terminating at the same vertex a and having the same label $(01)^\infty 111$. Therefore, $X \& Y$ is not AFT.

Now we give sufficient conditions such that the converse of Theorem 3.7 holds. Suppose X is a sofic shift with the Fischer cover $\mathcal{G} = (G, \mathcal{L})$. Let $G^{\#}$ be a new graph whose vertex set is the set $2^{\mathcal{V}}$ of subsets of the vertex set $\mathcal{V}$ of G. Let $\mathcal{A}$ be the alphabet of X. We draw an arrow labeled $a \in \mathcal{A}$ from a subset $F \in 2^{\mathcal{V}}$ to another subset $F' \in 2^{\mathcal{V}}$, when

$$F' = \{x \in \mathcal{V} : \text{ there is an edge labeled } a \text{ from an element of } F \text{ to } x\}.$$

We denote this new labeled graph by $(G^{\#}, \mathcal{L}^{\#})$. By [19, Proposition 6.5], $\partial X = \mathcal{L}^{\#}_\infty(X_{G_2^{\#}})$ where $G_2^{\#}$ denotes the subgraph of $G^{\#}$ obtained by erasing all vertices $F \in 2^{\mathcal{V}}$ for which $\#F \neq 2$, together with all arrows to or from such a vertex.

Lemma 3.9. *Let X be a sofic shift with the Fischer cover $\mathcal{G} = (G, \mathcal{L})$. Also let $x = p^\infty \in \partial X$ where p is primitive and let $p = \mathcal{L}(\pi_0)$ for some path π_0 in G. If there is only one cycle γ in G such that $x = \mathcal{L}_\infty(\gamma^\infty)$, then γ consists of concatinations of at least two copies of π_0.*

Proof. Let $p = p_0 p_1 \cdots p_{n-1}$. Then there is a cycle $\lambda = e_0 e_1 \cdots e_{n-1}$ in $G_2^{\#}$ such that $\mathcal{L}^{\#}(\lambda) = p$. Also suppose the edge e_i, $0 \le i \le n-1$ starts from the vertex $\{I_i, J_i\}$ and terminates at $\{I_{(i+1) \mod n}, J_{(i+1) \mod n}\}$. Note that if $e \in \mathcal{E}(G_2^{\#})$ starts from $\{K_1, L_1\}$ and terminates at $\{K_2, L_2\}$, since $K_i \neq L_i$ for $i = 1, 2$, e represents two

different edges e_1 and e_2 in G such that $i(e_i) \in \{K_1, L_1\}$ and $t(e_i) \in \{K_2, L_2\}$. So there are two paths π_1 and π_2 in G such that $\mathcal{L}(\pi_i) = \mathcal{L}(\pi_0) = p$ and

$$i(\pi_i),\, t(\pi_i) \in \{I_0,\, J_0\}, \quad i = 1,\, 2. \tag{3.3}$$

Suppose there is only one cycle γ in G such that $x = \mathcal{L}_\infty(\gamma^\infty)$. Since $I_0 \neq J_0$, I_0 and J_0 are different states along γ and by (3.3), they are initial and terminating points for two different paths in G labeled p and we are done. □

An immediate consequence of the above lemma is that if $x = p^\infty \in \partial X$, then there are two different paths π_1 and π_2 with $\mathcal{L}(\pi_i) = p$ for $i = 1, 2$ and either both are in a cycle γ or in the different cycles γ and γ' such that

$$p^\infty = \mathcal{L}_\infty(\gamma^\infty) = \mathcal{L}_\infty(\gamma'^\infty). \tag{3.4}$$

Lemma 3.10. *Suppose $\mathcal{G}$ is a finite right-resolving labeled graph with two different paths $\xi = \cdots e_{-1}e_0$, $\xi' = \cdots e'_{-1}e'_0$ and $\mathcal{L}(e_i) = \mathcal{L}(e'_i)$. Then there are two different cycles $e_{-m} \cdots e_{-n}$ and $e'_{-m} \cdots e'_{-n}$ in $\mathcal{G}$.*

Proof. There is at least one state v in G such that ξ meets it infinitely many often. Let $v = t(e_{-i_j})$ for $j \in \mathbb{N}$ and choose $j_m > |\mathcal{V}_G|$. Also let v'_j be the terminating state for e'_{-i_j}. We follow ξ and ξ' (backward) and simultaneously. Thus by pigeon principle, at least two states v'_{j_1} and v'_{j_2} amongst the j_m states $v'_1, \cdots, v'_{j_m}$ are equal and let $v' = v'_{j_1} = v'_{j_2}$. This means that when v' returns to itself along ξ', v returns to itself along ξ and so ξ and ξ' have met at least a cycle simultaneously on their ways. Call the cycles C_ξ and $C_{\xi'}$ respectively. Note that $C_\xi \neq C_{\xi'}$. Otherwise, there is a vertex w on the way of ξ and ξ' with two different outer edges lebeling the same which violates the fact that $\mathcal{G}$ is right-resolving. □

Theorem 3.11. *Let X and Y be two synchronized systems generated by $V = V_\alpha$ and $W = W_\beta$ as in 3.1 and $P_n(X) \cap P_n(Y) = \emptyset$ for all $n \in \mathbb{N}$. If $X = X_V$ and $Y = Y_W$ are SFT, AFT or near Markov, then $Z = X\&Y = X_V\&Y_W$ is SFT, AFT or near Markov, respectively.*

Proof. Suppose X and Y are SFT but Z is not so. Then $\partial X = \partial Y = \emptyset$ while $\partial Z \neq \emptyset$. Since ∂Z is a sofic subsystem of Z, there is a periodic point $p^\infty \in \partial Z$.

First suppose $\beta v\alpha \not\subseteq p$, for any $v\alpha \in V$. By the hypothesis, this means that either $p^\infty \in \mathcal{G}_{Z_X}$ or $p^\infty \in \mathcal{G}_{Z_Y}$. Suppose the former happens. Thus $\alpha \not\subseteq p^\infty$. Now choose m sufficiently large so that p^m is a synchronized word in X and $p^m \notin Y$. The existance of such m is guaranteed by the fact that X is SFT and $p^\infty \notin Y$. To have a contradiction, we show that p^m is a synchronized word for Z. So let up^m and $p^m w$ be arbitrary words for Z. Since $p^m \notin Y$, $u = u_1u'$ and $w = w'w_1$ where $u',\, w' \in \mathcal{B}(X)$ and they do not have α as a subword. We are trivially done if u_1 or w_1 is an empty word. Otherwise, without loss of generality assume $u_1 = \beta$ and $w_1 = \alpha$. Therefore, $\alpha u'p^m$ and $p^m w'\alpha$ are in X and this implies $\alpha u'p^m w'\alpha \in \mathcal{B}(X)$ and we are done.

Now suppose $\beta v\alpha \subseteq p$. Then $p = v_{i_1}\alpha w_{j_1}\beta \cdots v_{i_k}\alpha w_{j_k}\beta$ where $v_{i_r}\alpha \in V$ and $w_{i_r}\beta \in W$, $1 \leq r \leq k$. Without loss of generality assume that $p = v\alpha w\beta$ and let $V' = \{v : v\alpha \in V\}$, $W' = \{w : w\beta \in W\}$. If $v \notin W'$ (resp. $w \notin V'$), then $\beta v\alpha$ (resp. $\alpha w\beta$) is a sychronized word for Z and $p^\infty \notin \partial Z$. So $v,\, w \in V' \cap W'$ and by the definition of our generators

$$\alpha,\, \beta \not\subseteq v, \quad \alpha,\, \beta \not\subseteq w. \tag{3.5}$$

By Lemma 3.9, there are two different paths π_1 and π_2 in G_Z with $\mathcal{L}_Z(\pi_i) = p$ for $i = 1, 2$ and either both are in a cycle γ or in different cycles γ and γ' such that 3.4 holds. Consider the following cases.

(1) There are more than one cycle. Then (3.4) implies that $(v\alpha w\beta)^\infty = (v\beta w\alpha)^\infty$. By (3.5), either $v = w$ or $\alpha = \beta$. Considering the fact that any path labeled $v\alpha \in V$ (resp. $w\beta \in W$) terminates to the same state, the former will not allow $\mathcal{G}_Z$ being right-resolving and the latter contradicts our hypothesis $P_n(X) \cap P_n(Y) = \emptyset$ for all n.
(2) There is only one cycle γ with $p^\infty = \mathcal{L}_{Z_\infty}(\gamma^\infty)$. Then the label of this unique cycle γ must be $v\alpha w\beta$. But by Lemma 3.9, this cycle must be formed from the concatenation of at least two paths with the same label and (3.5) implies that in our situation $v\alpha = w\beta$ and this in turn implies $P_n(X) \cap P_n(Y) \neq \emptyset$ for some n.

As a result, $\partial Z = \emptyset$ and Z is SFT.

Suppose X and Y are AFT but Z is not AFT. So there are two different paths $\xi = \cdots e_{-1}e_0$ and $\xi' = \cdots e'_{-1}e'_0$ in G_Z with the same label and terminating at the same vertex. Also we may assume $e_0 \neq e'_0$ and let $C_\xi = e_{-m}\cdots e_{-n}$ and $C_{\xi'} = e'_{-m}\cdots e'_{-n}$ be two different cycles provided by Lemma 3.10.

(1) If C_ξ (resp. $C_{\xi'}$) is a cycle in G_{Z_X} (resp. G_{Z_Y}), then
$$\mathcal{L}(e_{-m}\cdots e_{-n}) = \mathcal{L}(e'_{-m}\cdots e'_{-n}) \in P_{n+m}(X) \cap P_{n+m}(X)$$
violating our hypothesis.
(2) If C_ξ and $C_{\xi'}$ are both cycles in G_{Z_X}, then we may assume that $t(e_0) = t(e'_0) = \mathbf{v}(\beta)$; otherwise, we may continue ξ and ξ' on a common path to get to $\mathbf{v}(\beta)$. But then we will have two different infinite paths labeled the same and terminating at the same vertex in $\mathcal{G}_X$ violating the fact that $\mathcal{G}_X^\alpha$ is left closing by Theorem 2.1.
(3) Note that in (1) and (2), $\mathcal{L}(C_\xi) = \mathcal{L}(C_{\xi'})$ does not have α or β as its subword. So the remaining case is that when $\alpha, \beta \subseteq \mathcal{L}(C_\xi)$. This implies $\mathcal{L}(C_\xi) = \mathcal{L}(C_{\xi'}) = w_{i_1}\beta v_{j_1}\alpha\cdots\alpha$. Let π_{i_1} and π'_{i_1} be the subpaths of C_ξ and C'_ξ such that $\mathcal{L}(\pi_{i_1}) = \mathcal{L}(\pi'_{i_1}) = w_{i_1}$. The fact that $\mathcal{G}_Z$ is right-resolving and paths labeled α terminate at the same vertex, implies that $\pi_{i_1} = \pi'_{i_1}$. By the same reasoning, paths in C_ξ and $C_{\xi'}$ labeled v_{j_1} are identical and carrying out this reasoning for all the subpaths of C_ξ and $C_{\xi'}$ we will have $C_\xi = C_{\xi'}$ which is absurd.

Now let X and Y be near Markov. So they are AFT and $|\partial X|, |\partial Y| < \emptyset$. Moreover, Z is AFT. If $|\partial Z| = \infty$, then since ∂Z is sofic there will be infinitely many periodic points in ∂Z. Apply the same reasoning as in the SFT – for the second part where $\beta v\alpha \subseteq p$ – to see that for any $p^\infty \in \partial Z \setminus (\partial X \cup \partial Y)$, we will have a contradiction. Thus $\partial Z \subseteq (\partial X \cup \partial Y)$ and we are done. □

A conclusion from our later Remark 5.6 is that no conclusion can be obtained for PFT between spaces and their intertwined system. That is there are PFT systems whose intertwined system is not PFT and also a PFT intertwined system emerging from two non-PFT systems.

4. (S, S')-gap Shifts

An important class of intertwined systems is the class of (S, S')-gap shifts mentioned in the introduction. In fact, S-gap shifts are symbolic dynamical systems which have many applications in practice, in particular, for coding of data [11] and in theory for its simplicity of producing different classes of dynamical systems. To define an S-gap shift $X(S)$, fix $S \subseteq \mathbb{N}_0$. If S is finite, define $X(S)$ to be the set of all binary sequences for which 1's occur infinitely often in each direction and such that the number of 0's between successive occurrences of a 1 is in S. When S is infinite, we need to allow points that begin or end with an infinite string of 0's.

The (S, S')-gap shifts may be considered as a generalization for S-gap shifts when the run of 0 and the run of 1 are restricted to S and S' as subsets of $\mathbb{N}_0$ respectively. In other words, fix two increasing sets S and S' in $\mathbb{N}_0$ and let $V = V_\alpha = \{v_s = 0^s1 : s \in S\}$ and $W = W_\beta = \{v_{s'} = 1^{s'}0 : s' \in S'\}$ where $\alpha = 1$ and $\beta = 0$. Then the intertwined system $Z = X(S, S')$ generated by $U = \{v_s v_{s'} : s \in S, s' \in S'\}$ is called a (S, S')-gap shift. An immediate observation is that $\mathcal{G}^\alpha_{X(S)} = \mathcal{G}_{X(S)}$ and $\mathcal{G}^\beta_{X(S')} = \mathcal{G}_{X(S')}$.

First we will give the Fischer cover for a sofic Z. Recall that the Fischer cover for any sofic shift is the labeled subgraph of the follower set graph consisting of only the follower sets of synchronizing words. Since for (S, S')-gap shifts, 10 and 01 are synchronizing words, 10^i and 01^i are synchronizing words for all $i \in \mathbb{N}$ [11, Lemma 3.3.15]. Also, for $v = v_1v_2...v_p$ (resp. $w = w_1w_2...w_q$) in $\mathcal{B}(Z)$ where $v_{p-1}v_p = 10$ (resp. $w_{q-1}w_q = 01$), we have $F(v) = F(10)$ (resp. $F(w) = F(01)$). On the other hand, 0^i and 1^i, $i \in \mathbb{N}$ are not synchronizing words. These facts can be used to establish the Fischer cover for (S, S')-gap shifts without resorting to the Fischer covers of the S-gap and S'-gap shifts as it was done for S-gap shifts in [2]. To avoid the tedious task of manipulating different cases, we use the Fischer covers of $X(S)$ and $X(S')$ and will obtain the Fischer cover for the associated (S, S')-gap shift. We begin by introducing an irreducible right-resolving presentation of Z, called $\mathcal{G}_Z$. By [11, Corollary 3.3.20] the merged graph of $\mathcal{G}_Z$ is the Fischer cover of Z.

We need to recall some facts about the Fischer cover of a S-gap shift. Set

$$\Delta(S) = \{d_1, d_2, \ldots, d_k, \overline{g_1, g_2, \ldots, g_l}\}, \qquad g = \sum_{i=1}^{l} g_i \tag{4.1}$$

where $d_1 = s_1$, $d_i = s_i - s_{i-1}$, $2 \le i \le k$ and $g_j = s_{k+j} - s_{k+j-1}, 1 \le j \le l$. Here k and l are the least integers such that (4.1) holds. Also

$$\mathcal{V}_S = \{F(1), F(10), \cdots, F(10^{n(S)})\}, \tag{4.2}$$

is the set of vertices where $n(S) = |S|$ for $|S| < \infty$.

Definition 4.1. If $|S| < \infty$, then set $n(S) = |S|$. If $|S| = \infty$, then $n(S)$ will be defined as follows.

(1) For $k = 1$ and $g_l > s_1$,
  (a) if $g_l = s_1 + 1$, then $F(10^{s_l+1}) = F(1)$ and $n(S) = s_l$.
  (b) if $g_l > s_1 + 1$, then $F(10^g) = F(1)$ and $n(S) = g - 1$.

(2) For $k \neq 1$, if $g_l > d_k$, then $F(10^{g+s_{k-1}+1}) = F(10^{s_{k-1}+1})$ and $n(S) = g + s_{k-1}$.

(3) For $k \in \mathbb{N}$, if $g_l \le d_k$, then $F(10^{s_{k+l-1}+1}) = F(10^{s_k - g_l + 1})$ and $n(S) = s_{k+l-1}$.

4.1. **Fischer cover for** $X(S,\, S')$**.** Suppose

$$S = \{s_i\}_{1\le i\le n(S)} \quad \text{and} \quad S' = \{s'_i\}_{1\le i\le n(S')} \tag{4.3}$$

are two increasing sequences in $\mathbb{N}_0$ and let $\mathcal{G}_S$ and $\mathcal{G}_{S'}$ be the Fischer covers of $X(S)$ and $X(S')$. Note that the main difference between a S-gap shift with a S'-gap shift is that the former restricts the run of 0 between two 1 whereas the latter restricts the run of 1 between two 0. So $\mathcal{V}_{\mathcal{S}} = \{F_S(1),\, F_S(10),\, \cdots,\, F_S(10^{n(S)})\}$ and $\mathcal{V}_{\mathcal{S}'} = \{F_{S'}(0),\, F_{S'}(01),\, \cdots,\, F_{S'}(01^{n(S')})\}$. Clearly the edges will be labled according to this fact.

Now we intertwine these two systems and will construct $\mathcal{G}_Z$ called the intertwined cover for $Z = X(S,\, S')$ (see Figure 3 for $|S|,\, |S'| < \infty$). Hence, we first replace the synchronizing word 1 (resp. 0) by 10 (resp. 01). This will change the vertices of G_S (resp. $G_{S'}$) from $F_S(10^i)$ (resp. $F_{S'}(01^j)$) to $F_Z(10^{i+1})$ (resp. $F_Z(01^{j+1})$), $0 \le i \le n(S)$, $0 \le j \le n(S')$. But we choose the terminating vertex of all edges labeled 1 (resp. 0) in G_S (resp. $G_{S'}$) be $F_Z(01)$ (resp. $F_Z(10)$) (see Figure 3). Therefore, G_Z and so $\mathcal{G}_Z = (G_Z,\, \mathcal{L}_Z)$ has been established.

As in (4.1), let $\Delta(S') = \{d'_1,\, d'_2, \ldots,\, d'_{k'}, \overline{g'_1,\, g'_2, \ldots,\, g'_{l'}}\}$ and $g' = \sum_{i=1}^{l'} g'_i$.

Theorem 4.2. *$\mathcal{G}_Z$ is the Fischer cover for $Z = X(S,\, S')$ if*

1. *S and S' are finite.*
2. *S is finite and S' does not satisfy (1b) in Definition 4.1.*
3. *S and S' are infinite and either S or S' does not satisfy (1a) in Definition 4.1.*

Proof. We need to show that $\mathcal{G}_Z$ is right-resolving and follower separated. To this end we have a labeled graph $\mathcal{G}_Z = (G_Z,\, \mathcal{L}_Z)$ which is an irreducible right-resolving presentation of Z but not necessarily minimal. The merged graph of $\mathcal{G}_Z$, say $\mathcal{H}$, is minimal and is the Fischer cover of Z. We show that $\mathcal{H} = \mathcal{G}_Z$ for our cases by showing that $\mathcal{G}_Z$ is follower separated. In fact we exclude cases where $\mathcal{G}_Z$ is not follower-separated. It is convenient to think of vertices of G_Z lying in two different rows, one consisting of $F_Z(10^i)$'s, $1 \le i \le n(S) + 1$ and the other $F_Z(01^j)$'s, $1 \le j \le n(S') + 1$.

Since the vertices in any row of G_Z are from the Fischer covers of $X(S)$ and $X(S')$, the follower sets representing vertices in the same row cannot be equal. Hence we look for those equal vertices in different rows. That is it is sufficient to consider the equality amongst a vertex $v = F(10^{s+1})$, $s \in S$ and $v' = F(01^{s'+1})$, $s' \in S'$.

(1)- Suppose S and S' are finite and let $S = \{s_1, \ldots,\, s_k\}$ and $S' = \{s'_1, \ldots,\, s'_{k'}\}$. Fix $1 \le i \le k$ and first let $i \ne k$ and set $v = F_Z(10^{s_i+1})$. Then v can be only equal to a $v' = F_Z(01^{s'_j+1})$ for some $s'_j \in S' \setminus \{s'_l\}$. But $0^{s_k+1} \in F_Z(01^{s'+1})$ for all $s' \in S' \setminus \{s'_l\}$ and 0^{s_k+1} is not in v as a follower set and consequently equality does not happen. Therefore, we let $i = k$ and we notice that in this case the out-degree of v is one with label 1. Thus v can only be equal to $v' = F_Z(01^{j_1+1})$ for some $j_1 \notin S'$. However, $1^{s'_l+1} \in F_Z(10^{s_k+1}) \setminus F_Z(01^{j+1})$ for all $j \notin S'$ and this implies $\mathcal{G}_Z$ is follower separated.

(2)- Suppose S is finite and $|S'| = \infty$. We show that unless $n(S')$ satisfies 1(b), then $\mathcal{G}_Z$ is follower-separated.

Suppose $v = v'$. Thus the out-degree and labelings of outer edges of v and v' must be the same and first, suppose their out-degree is one and call the associated

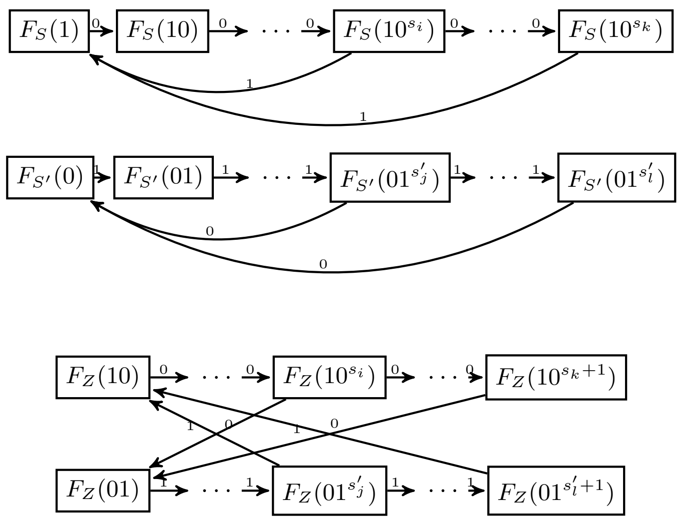

FIGURE 3. The Fischer cover of $X(S)$, $X(S')$ and $X(S,\,S')$.

edges e_v and $e_{v'}$ respectively. If $v = F_Z(10^i)$ (resp. $v' = F_Z(01^j)$), $1 \le i < n(S)+1$ (resp. $1 \le j \le n(S')+1$), then e_v (resp. $e_{v'}$) is labeled 0 (resp. 1) and if $v = F_Z(10^{n(S)+1})$, then e_v is labeled 1. Therefore to have $v = v'$, one must have v being the far right state in its row. Checking the possible cases, the situation where $|S| < \infty$, $|S'| = \infty$ and $X(S')$ satisfies (1b) is the only case with $v = v'$. In this situation, e_v and $e_{v'}$ both terminate at $F_Z(01)$ and are labeled 1 which implies both have the same follower sets and merging is required.

If the out-degree of v and v' is two, then $v = F_Z(10^{s+1})$ for $s \in S \setminus \{s_k\}$ and $v' = F_Z(01^{s'+1})$ for $s' \in S'$. But then $0^{s_k+1} \in F_Z(01^{s'+1}) \setminus F_Z(10^{s+1})$ and so $v \neq v'$.

(3)- Suppose both S and S' are infinite and assume $v = v'$. Recall that when $|S| = \infty$ (resp. $|S'| = \infty$), any edge starting from a vertex with the out-degree one, is labeled 0 (resp. 1). So the out-degrees of v and v' must be two. This means v (resp. v') is the starting state for an edge terminating at $F_Z(01)$ (resp. $F_Z(10)$) with label 1 (resp. 0) and another edge labeled 0 (resp. 1). Let $v = F_Z(10^{s_i+1})$. Then there is a path labeled $0^{s_j-s_i}1$, $i \le j \le k+l$ starting at v. This implies there exists a path with the same label starting at v' as well which contradicts the fact that k was the least integer. Thus $\mathcal{H} = \mathcal{G}_Z$ and no merging is needed. It remains to check for $v = F(10^{n(S)+1}) = F(01^{n(S')+1}) = v'$ and both with out-degree 2. In fact, for the case where both $X(S)$ and $X(S')$ satisfy (1a), $v = v'$: both v and v' have one edge with label 0 and terminating at the same vertex $F_Z(10)$ and one edge with label 1 and terminating at the same vertex $F_Z(01)$. Again merging is required. $\square$

From the above proof we have

Corollary 4.3. *If $\mathcal{G}_Z$ is not the Fischer cover for $Z = X(S,\,S')$, then $\mathcal{H}$ obtained from $\mathcal{G}_Z$ by merging $F_Z(10^{n(S)+1})$ and $F_Z(01^{n(S')+1})$ is the Fischer cover.*

4.2. **Adjacency Matrix for $X(S, S')$.** When S and S' are finite $Z = X(S, S')$ is SFT and so the adjacency matrix A_Z carries most of the information about the system. We use this matrix to give the Bowen-Franks groups of $X(S, S')$. One way to obtain A_Z is to use the presentation which is actually the Fischer cover for this case by Theorem 4.2. Doing so then A_Z is the $(n(S)+n(S')+2)\times(n(S)+n(S')+2)$ matrix

$$A_Z = \begin{pmatrix} A^1 & 0 \\ A^2 & A^3 \end{pmatrix} \tag{4.4}$$

where A^1 and A^3 are $(n(S)+1)\times(n(S)+2)$, $(n(S')+1)\times n(S')$ matrices and nonzero entries are as follows.

(1) $A^1_{i(i+1)} = A^1_{j(n(S)+2)} = 1$ for $1 \le i \le n(S)+1$, $j \in S \setminus \{n(S)+1\}$.
(2) $A^2_{(i+1)1} = 1$ for $i \in S'$.
(3) $A^3_{ii} = 1$ for $1 \le i \le n(S')$.

5. Dynamical Properties of (S, S')-gap Shifts

In this section we investigate the dynamical properties of (S, S')-gap shifts in terms of S and S'. We also give a formula for computing the entropy of these systems.

As we already have pointed out, a (S, S')-gap shift for $S' = \{0\}$ is in fact a $(S+1)$-gap shift. It is not surprising that most of the dynamical properties of (S, S')-gap shifts can be read from those of S-gap and S'-gap shifts. The following, however, shows that family of (S, S')-gap shifts are different from S-gap shifts.

Theorem 5.1. *There are infinitely many (S, S')-gap shifts which are not conjugate to any S-gap shift.*

Proof. Choose $|S|, |S'| = \infty$ so that $P_n(X(S, S')) \neq \emptyset$ and $P_{n+1}(X(S, S')) = \emptyset$ for some $n \in \mathbb{N}$ where P_n is the set of periodic points of period n (for instance, let $S = S' = k\mathbb{N}$, $k \in \mathbb{N}$). Also for $T \subseteq \mathbb{N}\cup\{0\}$ let $X(T)$ be a T-gap shift which is conjugate to $X(S, S')$. To arrive at a contradiction, we show that the set of periodic points of $X(S, S')$ and $X(T)$ are different. Since $|S|, |S'| = \infty$, $P_1(X(S, S')) = \{0^\infty, 1^\infty\}$ and so $0 \in T$. Furthermore, by conjugacy, $P_n(X(T)) \neq \emptyset$. Let $w^\infty \in P_n(X(T))$. Then $(1w)^\infty \in P_{n+1}(X(T))$ while $P_{n+1}(X(S, S')) = \emptyset$. $\square$

So far we know that (S, S')-gap shifts are all synchronized and so coded –a (S, S')-gap shift is generated by $\{0^{s+1}1^{s'+1} : s \in S, s' \in S'\}$. One can show directly that 10 and 01 are synchronizing words. By Theorem 2.2 and Theorem 2.3 we have

Theorem 5.2. *The following are equivalent for a (S, S')-gap shift Z.*

(1) *Z is mixing;*
(2) $\gcd\{s_n + s'_m + 2 : s_n \in S, s'_m \in S'\} = 1$*;*
(3) *Z is totally irreducible.*

Let $\mathcal{G} = (G, \mathcal{L})$ be a labeled graph, I a vertex of G and $A = A_G$ the associated adjacency matrix. The follower set $F_{\mathcal{G}}(I)$ of I in G is the collection of all labels of paths starting at I. The *period of an state* I, denoted by $\mathrm{per}(I)$, is the greatest common divisor of those integers $n \ge 1$ for which $(A^n)_{II} > 0$. The *period of the matrix* A denoted by $\mathrm{per}(A)$ is the greatest common divisor of all the numbers $\mathrm{per}(J)$ where J is a vertex. If A is irreducible, then all states have the same period.

The *period of a graph* G is the period of its adjacency matrix and is denoted by $\mathrm{per}(G)$. Let X_G be an irreducible edge shift and $p = \mathrm{per}(G)$. Then there exists a unique partition $\{D_0, D_1, ..., D_{p-1}\}$ of the vertices of G, called *period classes*, so that every edge that starts in D_i terminates in $D_{(i+1) \mod p}$.

Theorem 5.3. *Suppose $X(S, S')$ is a sofic shift with the Fischer cover $\mathcal{G} = (G, \mathcal{L})$. Then* $\mathrm{per}(G) = \gcd(S + S' + 2)$.

Proof. Let A be the adjacency matrix of G and $I = F(10)$. Starting at I on G, we are back again at I if we have traced on G a path labeled $0^{s_i}1$, $s_i \in S$ followed by a path labeled $1^{s'_j}0$, $s'_j \in S'$. Therefore,

$$\{n : (A^n)_{II} > 0, n \in \mathbb{N}\} = \left\{\sum(s_i + s'_j + 2) : s_i \in S, s'_j \in S'\right\}. \tag{5.1}$$

On the other hand,

$$\gcd\left\{\sum(s_i + s'_j + 2) : s_i \in S, s'_j \in S'\right\} = \gcd\{s + s' + 2 : s \in S, s' \in S'\}. \tag{5.2}$$

So (5.1) and (5.2) imply that $\mathrm{per}(I) = \gcd(S + S' + 2)$. Since A is irreducible, we are done. □

Other similarities of (S, S')-gap shifts with S-gap shifts is presented in the following theorem which characterizes some properties of $X(S, S')$ in terms of the combinatorial properties of S and S'; compare [10, Example 3.4] and [1, Theorems 3.3, 3.4 and 3.6]. Before, recall that $\Delta(S) = \{d_n\}_n$ and $\Delta(S') = \{d'_n\}_n$ where $d_1 = s_1$, $d'_1 = s'_1$, $d_n = s_n - s_{n-1}$ and $d'_n = s'_n - s'_{n-1}$, $n \geq 2$. Note that $\Delta(S)$ here is in consistence with (4.1).

Theorem 5.4. *$X(S, S')$ is*

(1) *SFT if and only if S and S' are finite or cofinite.*
(2) *sofic if and only if $\Delta(S)$ and $\Delta(S')$ are eventually periodic.*
(3) *AFT if and only if $\Delta(S)$ and $\Delta(S')$ are eventually constant.*
(4) *SVGL if and only if* $\sup_i |s_{i+1} - s_i| < \infty$ *and* $\sup_i |s'_{i+1} - s'_i| < \infty$.

Proof. The necessity and sufficient conditions for a S-gap shift being SFT, sofic and AFT was given in [1]. Also, a necessity and sufficient condition for a SVGL S-gap is in [10, Example 3.5]. Using those results, (1), (2), (4) and the necessity condition for (3) will be deduced from respective results in Theorem 3.7, Theorem 3.3 and Theorem 3.6. So it remains to prove the sufficiency of part (3). Suppose $\Delta(S)$ and $\Delta(S')$ are eventually constant. Then

$$\Delta(S) = \{d_1, d_2, ..., d_k, \overline{g}\}, \qquad \Delta(S') = \{d'_1, d'_2, ..., d'_{k'}, \overline{g'}\} \tag{5.3}$$

where $g = s_{k+1} - s_k$ and $g' = s'_{k'+1} - s'_{k'}$. A proof is established by showing that the intertwined cover of $X(S, S')$ is left-closing by giving a delay. The vertices that may prevent a labeled graph from being left-closing are those with more than one inner edges. Because, if it is not left-closing, then there are two paths ξ_- and ξ'_- labeled the same and terminating at the same vertex. Noticing the presentation of our $X(S, S')$ given in subsection 4.1, the vertices with more than one inner edges are $F(10)$, $F(01)$, $F(10^{(n(S)+2) \mod g})$ and $F(01^{(n(S')+2) \mod g'})$.

The vertex $F(10)$ (resp. $F(01)$) has k' (resp. k) inner edges such that $F(01^{s'_n+1})$ (resp. $F(10^{s_m+1})$) is the initial vertex of the nth edge, $1 \leq n \leq k'$ (resp. $1 \leq m \leq k$). The label of a path of length $s'_{k'} + 2$ (resp. $s_k + 2$) terminating at $F(10)$

(resp. $F(01)$) determines its ending edge. On the other hand, The label of a path of length $(n(S)+2)(\mod g)+1$ (resp. $(n(S')+2)(\mod g')+1$) terminating at $F(10^{(n(S)+2) \mod g})$ (resp. $F(01^{(n(S')+2) \mod g'})$) determines its ending edge. So the intertwined cover has $D = \max\{s_k+2,\, s'_{k'}+2,\, (n(S)+2)(\mod g)+1,\, (n(S')+2)(\mod g')+1\}$ as its delay and the proof is complete. $\square$

Example 3.8 was for two S-gap shifts which shows that Theorem 5.4 is not necessarily true if we choose different synchronizing words.

Theorem 5.5. *Any AFT $(S,\, S')$-gap shift is near Markov.*

Proof. Let $Z = X(S,\, S')$ be a strictly AFT shift. Two words 10 and 01 are synchronizing words. Hence, points having both 0 and 1 as some of its entries are not in ∂Z and $\partial Z \subseteq \{0^\infty, 1^\infty\}$. So ∂Z is finite. $\square$

Remark 5.6. Not all the properties of $X(S)$ and $X(S')$ transfer to $Z = X(S,\, S')$ or vice versa. Mixing and PFT are two such properties. As an example, let $S = S' = 2\mathbb{N}$ and $\mathcal{G}_X = (G_X, \mathcal{L}_X)$ be the Fischer cover of X. Then both $X(S)$ and $X(S')$ are mixing [10] while $\gcd(S+S') = 2$ and so Z does not have mixing property by Theorem 5.2. Also the same S and S' as above will imply that Z is PFT. This is a consequence of the fact that $\mathrm{per}(G_Z) = 2$ and the fact that the irreducible components of $\mathcal{G}_Z$ and $\mathcal{G}_Z^2$ are definite graphs [4, Proposition 8]. On the other hand, since $\mathrm{per}(\mathcal{G}_{X(S)}) = \mathrm{per}(\mathcal{G}_{(X(S')}) = 1$, $X(S)$ and $X(S')$ are not PFT [14, Proposition 1].

Now set $S = 3\mathbb{N} - 1$ and $S' = 5\mathbb{N} - 1$. Then

$$\gcd\{s+1 : \, s \in S\} = 3, \qquad \gcd\{s'+1 : \, s' \in S'\} = 5$$

which means $X(S)$ and $X(S')$ are not mixing [10] and since $\Delta(S) = \{2, \overline{3}\}$ and $\Delta(S') = \{4, \overline{5}\}$, both $X(S)$ and $X(S')$ are PFT [1, Theorem 3.8]. However, $\gcd(S+S'+2) = 1$ and so Z is mixing (Theorem 5.2) but not PFT [14, Proposition 1].

Recall that a S-gap shift is strictly *PFT* if and only if it is AFT and non-mixing [1, Theorem 3.8]. We give the same characterization for $(S,\, S')$-gap shifts which are strictly PFT.

Theorem 5.7. *Suppose $X = X(S,\, S')$ is not SFT. Then it is strictly PFT if and only if it is AFT and non-mixing.*

Proof. Suppose $Z = X(S,\, S')$ is strictly PFT. Since Z is irreducible; it must be AFT [14, Theorem 2]. If it is mixing, then $\mathrm{per}(A_{\mathcal{G}}) = 1$ [11, Exercise 4.5.16] where $\mathcal{G}$ is the Fischer cover of Z and this contradicts [14, Proposition 1].

To prove the necessity, suppose Z is AFT and non-mixing and suppose $\mathcal{G} = (G, \mathcal{L})$ is the Fischer cover of Z. Let $p = \mathrm{per}(A_{\mathcal{G}})$ and let $D_0, D_1, ..., D_{p-1}$ be the period classes of $\mathcal{G}$. Since every path with label 10 or 01 terminates at $F(10)$ or $F(01)$ respectively, so the initial vertices of these paths are in the same period class. Without loss of generality, assume that it is D_0. Then it is obvious that $\mathcal{H} = \{(10)^i, (01)^i : \, i \in \{\{1, \ldots, p-2,\, p-1\}\} \subseteq \mathcal{O}$ where $\mathcal{O}$ is the collection of all periodic first offenders. To prove the theorem, we show that $\mathcal{O} = \mathcal{H}$ and hence finite. Suppose $w^{(n)} = (w_0, w_1, ..., w_{l-1})^{(n)} \in \mathcal{O} \backslash \mathcal{H}$. Then $w \notin \cup_{I \in D_n} F_{\mathcal{C}}(I)$ for some $0 \le n \le p-1$. So $w \notin \{0^l,\, 1^l\}$. Now if $w_i w_{i+1} = 10$ where $0 \le i \le l-2$, then $10 \notin D_{(n+i) \mod p}$ which implies $w \notin \mathcal{O}$. $\square$

5.1. **Entropy of $(S,\,S')$-gap Shifts.** Let S and S' be the subsets of $\mathbb{N}_0$. If the multiplicity in $S+S'$ is important we will show it by $\{\!\{S+S'\}\!\}$. Thus, if $S=\{1,\,3\}$ and $S'=\{2,\,4\}$, then $S+S'=\{3,\,5,\,7\}$ but $\{\!\{S+S'\}\!\}=\{\!\{3,\,5,\,5,\,7\}\!\}$. When no multiplicities exists, we write $S+S'=\{\!\{S+S'\}\!\}$. When S and S' are finite, the number of elements in $\{\!\{S+S'\}\!\}$ is $|S||S'|$. We will see that $\{\!\{S+S'\}\!\}$ is a conjugacy invariant and in particular characterizes the entropy.

The entropy of a shift space X is defined by $h(X)=\lim_{n\to\infty}(1/n)\log|\mathcal{B}_n(X)|$. This reduces to a more computable formula when X has SVGL. First let X be synchronized and fix a synchronizing word $w\in\mathcal{B}(X)$. Let $C_n(X)$ be the set of words $v\in\mathcal{B}_n(X)$ such that $wvw\in\mathcal{B}(X)$. Then the synchronized entropy $h_{\mathrm{syn}}(X)$ is defined by

$$h_{\mathrm{syn}}(X)=\limsup_{n\to\infty}\frac{1}{n}\log|C_n(X)|.$$

This value is independent of w and $h(X)\geq h_{\mathrm{syn}}(X)$. In general, $h(X)\neq h_{\mathrm{syn}}(X)$, however, Thomsen [19] showed that for irreducible sofic shifts $h(X)=h_{\mathrm{syn}}(X)$. Later Jung [10] extended this result to SVGL shifts. We exploit this to compute the entropy for $(S,\,S')$-gap shifts.

Theorem 5.8. *The entropy of a $(S,\,S')$-gap shift is $\log\lambda$ where λ is the unique non-negative solution of*

$$\sum_{s+s'\in\{\!\{S+S'\}\!\}}x^{-(s+s'+2)}=\sum_{n\in S+S'}\left(\sum_{k+l=n}\chi_S(k)\chi_{S'}(l)\right)x^{-(n+2)}=1. \tag{5.4}$$

Proof. Since $Z=X(S,\,S')$ is a synchronized system, we will compute $h_{\mathrm{syn}}(Z)$ and we will choose our synchronizing word to be $w=01$. Therefore, by setting $C_i=C_i(Z)$, we have

$$\begin{aligned}
C_1 &= \chi_S(1)\chi_{S'}(0)+\chi_S(0)\chi_{S'}(1),\\
C_2 &= \chi_S(2)\chi_{S'}(0)+\chi_S(0)\chi_{S'}(2)+\chi_S(1)\chi_{S'}(1)+\chi_{S\cap S'}(0),\\
C_3 &= \chi_S(3)\chi_{S'}(0)+\chi_S(0)\chi_{S'}(3)+\chi_S(2)\chi_{S'}(1)+\chi_S(1)\chi_{S'}(2)+\\
&\quad \chi_{S\cap S'}(0)C_1+\chi_{S\cap S'}(0)(\chi_S(1)+\chi_{S'}(1)).
\end{aligned}$$

By an induction arguement on n, we will have,

$$\begin{aligned}
C_n &= \sum_{k+l=n}\chi_S(k)\chi_{S'}(l)+\sum_{p\in S+S'}\left(\sum_{k+l=p}\chi_S(k)\chi_{S'}(l)\right)C_{n-(p+2)}+\\
&\quad \chi_{S\cap S'}(0)\left(\chi_S(n-2)+\chi_{S'}(n-2)+O(n)\right).
\end{aligned}$$

In other words,

$$\begin{aligned}
1 &= \sum_{k+l=n}\frac{\chi_S(k)\chi_{S'}(l)}{C_n}+\sum_{p\in S+S'}\left(\sum_{k+l=p}\chi_S(k)\chi_{S'}(l)\right)\frac{C_{n-(p+2)}}{C_n}+\\
&\quad \frac{\chi_{S\cap S'}(0)}{C_n}\left(\chi_S(n-2)+\chi_{S'}(n-2)+O(n)\right). \qquad (1.1)
\end{aligned}$$

Now for some subsequence $\{n_t\}_{t\in\mathbb{N}}$, C_{n_t} is asymptotic to λ^{n_t} as $t\to\infty$ where $\lambda=2^{h_{\mathrm{syn}}(Z)}>1$. Therefore,

$$\lim_{t\to\infty}\frac{C_{n_t-(p+2)}}{C_{n_t}}=\lambda^{-(p+2)}.$$

By letting $t \to \infty$ in (1.1), $\sum_{p\in S+S'}\left(\sum_{k+l=p}\chi_S(k)\chi_{S'}(l)\right)\frac{1}{\lambda^{p+2}} = 1$. So, $h_{\rm syn}(Z) = \log\lambda$ where λ is the non-negative solution of the equation

$$\sum_{s+s'\in\{\!\{S+S'\}\!\}} x^{-(s+s'+2)} = 1.$$

If Z has SVGL, then $h(Z) = h_{\rm syn}(Z)$ [10]. Therefore, suppose Z does not have such a property. Let $S_n = \{s_1,\, s_2, \cdots, s_n\}$, $S'_n = \{s'_1,\, s'_2, \cdots, s'_n\}$, $n \in \mathbb{N}$. If one of S or S', say S, is finite, then there is a $N \in \mathbb{N}$ such that $S_n = S$ for all $n \geq N$. Then $h(X(S_n, S'_n)) = h_{\rm syn}(X(S_n,\, S'_n))$ and the sequence $\{h(X(S_n, S'_n))\}_n$ is an increasing sequence with upper bound $h(Z)$. First we show that $h(X(S_n, S'_n)) \longrightarrow h(Z)$. Let λ_n be the Perron eigenvalue of $h(X(S_n,\, S'_n))$. We have

$$\lim_{n\to\infty}\log\lambda_n = \lim_{n\to\infty} h(X(S_n, S'_n)) = \lim_{n\to\infty}\lim_{m\to\infty}\frac{1}{m}\log\mathcal{B}_m(X(S_n,\, S'_n)).$$

For any fixed m, $\mathcal{B}_m(X(S_n, S'_n))$ increases to $\mathcal{B}_m(Z)$. So

$$\begin{aligned}\lim_{n\to\infty}\lim_{m\to\infty}\frac{1}{m}\log\mathcal{B}_m(X(S_n,\, S'_n)) &= \lim_{m\to\infty}\lim_{n\to\infty}\frac{1}{m}\log\mathcal{B}_m(X(S_n,\, S'_n))\\ &= \lim_{m\to\infty}\frac{1}{m}\log\mathcal{B}_m(Z) = h(Z).\end{aligned}$$

If we let $h(Z) = \log\lambda$ for some $\lambda > 0$, then $\lim_{n\to\infty}\lambda_n = \lambda$. Now set

$$f_n(x) = \sum_{s_i+s'_j\in\{\!\{S_n+S'_n\}\!\}} x^{-(s_i+s'_j+2)} - 1, \qquad f(x) = \sum_{s+s'\in\{\!\{S+S'\}\!\}} x^{-(s+s'+2)} - 1.$$

Note that $f_n(x)$ and $f(x)$ are decreasing functions on x and for any fixed x, $\{f_n(x)\}_n$ increases to $f(x)$. So λ_n, the unique solution of $f_n(x) = 0$, converges to the unique solution of $f(x) = 0$. This means $f(\lambda) = 0$ and we are done. $\square$

For a dynamical system (X, T), let p_n be the number of periodic points in X having period n. When $p_n < \infty$, the zeta function $\zeta_T(t)$ is defined as

$$\zeta_T(t) = \exp\left(\sum_{n=1}^{\infty}\frac{p_n}{n}t^n\right). \tag{5.5}$$

Then by Taylor's formula,

$$\frac{d^n}{dt^n}\log\zeta_T(t)|_{t=0} = n!\frac{p_n}{n} = (n-1)!p_n. \tag{5.6}$$

Theorem 5.9. *If $\zeta_{\sigma(S,\,S')}(X(S,\, S')) = \zeta_{\sigma(T,\,T')}(X(T,\, T'))$, then $\{\!\{S+S'\}\!\} = \{\!\{T+T'\}\!\}$. When $\{\!\{S+S'\}\!\} = \{\!\{T+T'\}\!\}$, then $h(X(S,\, S')) = h(X(T,\, T'))$.*

Proof. Let $\{\!\{S+S'\}\!\} = \{\!\{v_1, v_2, \cdots\}\!\}$ with $v_i \leq v_{i+1}$, $i \geq 1$. For $n = 1$, $v_1 = s_1 + s'_1 = t_1 + t'_1 \in \{\!\{T+T'\}\!\}$. Assume for any $n < N$, $v_n = t_m + t'_{m'} \in \{\!\{T+T'\}\!\}$ for some $(t_m,\, t'_{m'}) \in T\times T'$ and consider $v_N = s_{n_0} + s'_{n'_0}$, $(s_{n_0},\, s'_{n'_0}) \in S\times S'$.

By equality of zeta functions and (5.6), we have that for all i, $p_i(X(S,\, S')) = p_i(X(T,\, T'))$; in particular,

$$p_{v_N}(X(S,\, S')) = p_{v_N}(X(T,\, T')). \tag{5.7}$$

Now if $(s_{n_0} + s'_{n'_0}) \notin \{\!\{T+T'\}\!\}$, then (5.7) implies that $s_{n_0} + s'_{n'_0} = \sum_{r=1}^{p}(t_{u_r} + t'_{u'_r})$ with $p > 1$ and $(t_{u_r} + t'_{u'_r}) < v_N$. But any periodic point of period n of $X(S,\, S')$ looks like

$$(1^{s'_{i_1}+1}0^{s_{j_1}+1}1^{s'_{i_2}+1}0^{s_{j_2}+1}\cdots 1^{s'_{i_q}+1}0^{s_{j_q}+1})^{\infty} \tag{5.8}$$

where $\sum_{r=1}^{q}(s'_{i_r}+s_{j_r}+2)=n$. Thus by our induction assumption, there is a one to one correspondence between those elements of $\{\!\{S+S'\}\!\}$ and $\{\!\{T+T'\}\!\}$ whose values are less than v_N. As a result, we must have $p_{v_N}(X(S,\,S'))\geq p_{v_N}(X(T,\,T'))+1$ which contradicts (5.7). $\square$

A shift space X is called *almost sofic* if $h(X)=\sup\{h(Y):Y\subseteq X$ is a sofic subshift$\}$ [17, Definition 6.7].

Theorem 5.10. *Every $(S,\,S')$-gap shift is almost sofic.*

Proof. If $X(S,\,S')$ is sofic, the statement is obvious. Thus suppose $X(S,\,S')$ is not sofic and for every $k\geq 1$, define $S_k=\{s_1,s_2,...,s_k\}$ and $S'_k=\{s'_1,s'_2,...,s'_k\}$. Then for all k, $X(S_k,\,S'_k)$ is a sofic subsystem of $X(S,\,S')$ and $\{h(X(S_k,\,S'_k))\}_{k\geq 1}$ is an increasing sequence. By (5.4), $h(X(S_k,\,S'_k))\nearrow h(X(S,\,S'))$ which implies $X(S,\,S')$ is almost sofic. $\square$

A $(S,\,S')$-gap shift is synchronized; so their nontrivial subshift factors are coded [5] and with positive entropy [12, Proposition 2.12]. Moreover, next theorem shows that every subshift factor of a $(S,\,S')$-gap shift is *intrinsically ergodic*: there is a unique measure of maximal entropy. This fact was established for S-gap shifts by Climenhaga and Thompson in [6].

Theorem 5.11. (1) *Every subshift factor of a $(S,\,S')$-gap shift is intrinsically ergodic.*
(2) *If at least one of the S or S' is infinite and Y is a subshift factor of $X(S,\,S')$, then either Y is nontrivial or $Y=\{a^\infty\}$.*

Proof. (1). Let

$$\mathcal{L}_1=\{0^k1^{m_1}0^{n_1}\cdots 1^{m_p}0^{n_p}1^l:\,m_i-1\in S';n_i-1\in S,k,l\in\mathbb{N}\},$$

$$\mathcal{L}_2=\{1^{k'}0^{n'_1}1^{m'_1}0^{n'_1}\cdots 0^{n'_q}1^{m'_q}0^{l'}:\,m'_j-1\in S';n'_j-1\in S,k,k',l'\in\mathbb{N}\}.$$

Then language of $(S,\,S')$-gap shift is $\mathcal{L}_1\cup\mathcal{L}_2$. So a *uniform CGC-decomposition* for $X(S,S')$ is given by

$$\begin{aligned}\mathcal{G}&=\{0^n1^m:\,n-1\in S,m-1\in S'\},\\ \mathcal{C}^p&=\{0^k,1^k:\,k\geq 0\},\\ \mathcal{C}^s&=\{0^l,1^l:\,l\geq 1\}.\end{aligned}$$

Let $(\mathcal{C}^p\cup\mathcal{C}^s)_n$ be the words in $\mathcal{C}^p\cup\mathcal{C}^s$ of length n. Then $|(\mathcal{C}^p\cup\mathcal{C}^s)_n|=2$ for all $n\geq 1$ and so $h(\mathcal{C}_p\cup\mathcal{C}_s)=0$. Therefore, it follows that every subshift factor of a $(S,\,S')$-gap shift is intrinsically ergodic [6].

(2). In [6], Climenhaga and Thompson proved that any shift with uniform CGC-decomposition, either has positive entropy or comprises a single periodic orbit and they also proved that a factor shift of a shift with uniform CGC-decomposition has uniform CGC-decomposition. Now suppose Y is a shift space over the alphabet $\mathcal{A}$ and let $\phi:X(S,\,S')\to Y$ be a factor code with memory m and anticipation n induced by the block map Φ and suppose $|S|=\infty$. Then $\Phi(0^{m+n+1})$ is a symbol in $\mathcal{A}$, say a. Since $|S|=\infty$, the language of Y must contain a^k for any $k\in\mathbb{N}$. This means Y is either nontrivial or $Y=\{a^\infty\}$. $\square$

6. The Bowen-Franks Groups of $(S,\, S')$-gaps

Let A be a $n \times n$ integer matrix. The *Bowen-Franks group* of A is

$$BF(A) = \mathbb{Z}^n/\mathbb{Z}^n(\mathrm{Id} - A),$$

where $\mathbb{Z}^n(\mathrm{Id} - A)$ is the image of $\mathbb{Z}^n$ under the matrix $\mathrm{Id} - A$ acting on the right. In order to compute the Bowen-Franks group, we will use the Smith form defined below for an integral matrix. Define the *elementary operations* over $\mathbb{Z}$ on integral matrices to be:

(1) Exchanging two rows or two columns.
(2) Multiplying a row or column by -1.
(3) Adding an integer multiple of one row to another row, or of one column to another column.

Every integral matrix can be transformed by a sequence of elementary operations over $\mathbb{Z}$ into a diagonal matrix

$$\begin{pmatrix} d_1 & 0 & 0 & \cdots & 0 \\ 0 & d_2 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & d_n \end{pmatrix}$$

where $d_j \geq 0$ and d_j divides d_{j+1}. This is called the *Smith form* of the matrix [15]. If we put $\mathrm{Id} - A$ into its Smith form, then

$$BF(A) \simeq \mathbb{Z}_{d_1} \oplus \mathbb{Z}_{d_2} \oplus \ldots \oplus \mathbb{Z}_{d_n}.$$

By our convention, each summand with $d_j = 0$ is $\mathbb{Z}$, while summands with $d_j > 0$ are finite cyclic groups. Since $\mathbb{Z}_1$ is the trivial group, the elementary divisors of $BF(A)$ are the diagonal entries of the Smith form of $\mathrm{Id} - A$ which are not 1.

Note that $BF(A)$ (or denoted by $BF_0(A)$ in some papers) is the cokernel of $\mathrm{Id} - A$ acting on the row space $\mathbb{Z}^n$. The kernel is another Bowen-Franks group $BF_1(A) := \mathrm{Ker}(\mathrm{Id} - A)$. Similarly, acting on the column space $(\mathbb{Z}^n)^t$, $\mathrm{Id} - A$ defines another two groups as cokernel and kernel, denoted by $BF^t(A)$ and $BF^t_1(A)$ respectively. These four groups are called the BF-groups [9].

Theorem 6.1. *Let $X(S,\, S')$ be a SFT shift.*

(1) *Suppose $|S|, |S'| < \infty$. Then $BF(A) \simeq \mathbb{Z}_{(|S||S'|-1)} \simeq BF^t(A)$.*
(2) *Suppose $|S| < \infty$ and $|S'| = \infty$. Then $BF(A) \simeq \mathbb{Z}_{|S|} \simeq BF^t(A)$.*
(3) *Suppose $|S|, |S'| = \infty$. Then $BF(A) \simeq \mathbb{Z}_1 \simeq BF^t(A)$.*
(4) *$BF_1(A) = BF^t_1(A) = \{0\}$.*

Proof. The adjacency matrix of the Fischer cover of $X(S,\, S')$ is obtained in Subsection 4.2. Then $BF_1(A) = BF^t_1(A) = \{0\}$ is a consequence of the definition. Groups $BF(A)$ and $BF^t(A)$ can be found by computing the Smith forms of $\mathrm{Id} - A$ and $\mathrm{Id} - A^t$ respectively. □

Two subshifts are flow equivalent if they have topologically equivalent suspension flows [11]. Franks in [7] classified irreducible SFT's up to flow equivalent by showing that two nontrivial irreducible SFT's X_A and X_B are flow equivalent if and only if $BF(A) \simeq BF(B)$ and $\mathrm{sgn}(\det(\mathrm{Id} - A)) = \mathrm{sgn}(\det(\mathrm{Id} - B))$.

Corollary 6.2. *Suppose $X(S,\, S')$ is a SFT shift.*

(1) *If* $|S|, |S'| < \infty$, *then* $X(S,\, S')$ *is flow equivalent to full* $|S||S'|$*-shift.*
(2) *If* $|S| < \infty$ *and* $|S'| = \infty$, *then* $X(S,\, S')$ *is flow equivalent to full* $(|S|+1)$*-shift.*
(3) *If* $|S|, |S'| = \infty$, *then* $X(S,\, S')$ *is flow equivalent to full* 2*-shift.*

The Bowen-Franks groups for a RLL shift was announced in [2]. By above corollary, for an asymmetric-RLL $X = X(d_1,\, k_1,\, d_0,\, k_0)$ constraint with the adjacency matrix A, $BF(A) \simeq \mathbb{Z}_{(k_0-d_0+1)(k_1-d_1+1)-1} \simeq BF^t(A)$ and X is flow equivalent to full $(k_0 - d_0 + 1)(k_1 - d_1 + 1)$-shift.

7. The Conjugacy Problem for $(S,\, S')$-gap Shifts

The conjugacy problem has been solved completely for S-gap shifts [1]. Here, we give a general necessary condition and some sufficient conditions for special cases.

The following result is a consequence of Theorem 5.9.

Theorem 7.1. *If* $X(S,\, S')$ *and* $X(T,\, T')$ *are conjugate, then* $\{\!\{S+S'\}\!\} = \{\!\{T+T'\}\!\}$.

This is by no means a conjugacy. For instance let $S = \{1, 2\}$, $S' = 2\mathbb{N}_0 + 1$, $T = \{1\}$ and $T' = \{1, 2, \cdots\}$. Then $\{\!\{S + S'\}\!\} = \{\!\{T + T'\}\!\}$. By 5.4, $h(X(S,\, S')) = h(X(T,\, T'))$. But $X(S,\, S')$ is a strictly sofic shift while $X(T,\, T')$ is a SFT shift, so they are not conjugate.

Also,

$$\zeta_{X(S,\, S')}(t) = \zeta_{X(T,\, T')}(t) = \frac{1}{1 - t - t^4}.$$

Thus unlike S-gap shifts [1, Corollary 4.2], the zeta function is not a complete invariant for conjugacy.

Theorem 7.2. *If* $\{\!\{S+S'\}\!\} = \{\!\{T+T'\}\!\}$, *then* $X(S,\, S')$ *and* $X(T,\, T')$ *are conjugate in the following cases.*

(1) $s_i + s'_j = t_i + t'_j$ *for all* i *and* j,
(2) $X(S,\, S')$ *and* $X(T,\, T')$ *are SFT and* $S+S' = \{\!\{S+S'\}\!\} = \{\!\{T+T'\}\!\} = T+T'$. *In particular, they are conjugate to a* S*-gap shift* $X(S'')$, $S'' = S+S'+1$.

Proof. (1). Without loss of generality assume $s'_1 > t'_1$. Let $n = s'_1 - t'_1 + 1$ and define $\Phi : \mathcal{B}_n(X(S,\, S')) \to \{0, 1\}$,

$$\Phi(w) = \begin{cases} 1 & w = 1^n, \\ 0 & \text{otherwise.} \end{cases}$$

Then $\phi = \Phi_\infty^{[0,\, n-1]} : X(S,\, S') \to X(T,\, T')$ is a conjugacy map. In fact, what ϕ does is to replace $1^{s'_i+1}0^{s_j+1}1^n$ with $1^{t'_i+1}0^{t_j+1}$. This is because the word $1^{s'_i+1}$ contains $(s'_i+1-(n-1))$ times the word 1^n; the first starting from the first position of $1^{s'_i+1}$, the second from the second position and so forth. But $s'_i+1-(n-1) = s'_i-s'_1+t'_1+1$ and also by (1),

$$s_1 + s'_i = t_1 + t'_i, \qquad s_1 + s'_1 = t_1 + t'_1.$$

So $s'_i - s'_1 + t'_1 = t'_i$. By the same reasoning, $(n-1) + s_j + 1 = t_j + 1$ and so we are done.

(2). A conjugacy map can be set via a S-gap shift $X(S'')$ as follows. First define $\Phi : \mathcal{B}_2(X(S,\, S')) \to \{0, 1\}$,

$$\Phi(w) = \begin{cases} 1 & w = 10, \\ 0 & \text{otherwise.} \end{cases} \tag{7.1}$$

Now let $S'' = S + S' + 1$ and $\phi : X(S,\, S') \to X(S'')$ be the sliding block code with memory 0 and anticipation 1 induced by Φ. Then ϕ defines a conjugacy map. Also, $X(T,\, T')$ and $X(S'')$ are conjugate with the same conjugacy map by letting $\Phi : \mathcal{B}_2(X(T,\, T')) \to \{0, 1\}$ be defined as (7.1). $\square$

Since $\{\!\{S + S'\}\!\} = |S||S'|$ the following is immediate.

Corollary 7.3. *If $\{\!\{S + S'\}\!\} = \{\!\{T + T'\}\!\}$ and $|\{\!\{S + S'\}\!\}| = p$ for a prime $p \in \mathbb{N}$, then $X(S,\, S')$ and $X(T,\, T')$ are conjugate to the S-gap shift $X(S + S' + 1)$ and so conjugate to each other.*

Which $(S,\, S')$-gap shifts, as a generalization for the Run-length-limited (RLL) shifts, are conjugate to a given RLL shift?

Although the problem of conjugacy for $(S,\, S')$-gap shifts has not been sorted out for general settings, but in most cases there are several $(S,\, S')$-gap shifts which are conjugate to a given RLL shift. Let $|S| = p$ and $|S'| = p'$. If $X(S,\, S')$ and a RLL shift $X(d,\, k)$ are conjugate, then by Theorem 7.1, $\{\!\{S + S'\}\!\} = \{d - 1,\, d - 2, \dots,\, k - 1\}$. So we have

$$d = s_1 + s'_1 + 1, \quad \text{and} \quad k = s_p + s'_{p'} + 1 \tag{7.2}$$

where $s_1,\, s'_1$ are minimum and $s_p,\, s'_{p'}$ are maximum of the respective spaces.

Now suppose d and k are the natural numbers such that $k > d$ and $k - d + 1$ is not prime. Also set $k - d + 1 = p \times p'$, $1 < p \leq p'$. Then RLL shift $X(d,\, k)$ and $X(S,\, S')$ are conjugate for some

$$S = \{s_1,\, s_2, \cdots,\, s_p\}, \quad \text{and} \quad S' = \{s'_1,\, s'_2, \cdots,\, s'_{p'}\} \tag{7.3}$$

satisfying (7.2) and

$$s_{i+1} - s_i = p', \qquad s'_{j+1} - s'_j = 1$$

for $1 \leq i \leq p - 1$ and $1 \leq j \leq p' - 1$. The conjugacy follows from the case (2) of Theorem 7.2. So other $(S,\, S')$-gap shifts conjugate to $X(d,\, k)$ may be found when $\{\!\{S + S'\}\!\}$ equals $\{d - 1,\, d - 2, \dots,\, k - 1\}$.

Corollary 7.4.
1. *If $X(S,\, S')$ and a RLL shift $X(d,\, k)$ are conjugate, then S and S' are SFT with $|S| = p$ and $|S'| = p'$ for some $p,\, p' \in \mathbb{N}$ and d and k satisfies (7.2).*
2. *Suppose $X(d,\, k)$ is a RLL shift with $k - d + 1 = p \times p'$, $1 < p \leq p'$. Then $X(d,\, k)$ and $X(S,\, S')$ are conjugate for some S and S' as in (7.3).*
3. *Let $p \in \mathbb{N}$ be a prime number and $X(d,\, k)$ be a RLL shift with $k - d + 1 = p$. Then $X(d,\, k)$ is conjugate to a $X(S,\, S')$, with $S = \{s\}$, $s \leq d - 1$ and $S' = \{d - s - 1, d - s, \dots, k - s - 1\}$.*

Example 7.5. The RLL shift $X(2,\, 9)$ and $X(S,\, S')$ are conjugate for

1. $S = \{1,\, 5\}$, $S' = \{1,\, 2,\, 3,\, 4\}$,
2. $S = \{1,\, 2\}$, $S' = \{1,\, 3,\, 5,\, 7\}$,
3. $S = \{1,\, 3\}$, $S' = \{1,\, 2,\, 5,\, 6\}$.

The first two are implied by Corollary 7.4(2) and the third where S and S' do not satisfy the conditions of S and S' in 7.3 is a consequence of Theorem 7.2(2).

References

1. D. Ahmadi Dastjerdi and S. Jangjoo, *Dynamics and topology of S-gap shifts*, Topology and its Applications **159** (2012) 2654–2661.
2. D. Ahmadi Dastjerdi and S. Jangjoo, Computations on sofic S-gap shifts, arXiv: 1108.3414v5.
3. J. Banks, Regular periodic decompositions for topologically transitive maps, *Discrete & Cont. Dynam. Sys.* **5** (1997), 83-92.
4. M. -P. Beal, M. Crochemore, B. E. Moision and P. H. Siegel, Periodic finite-type shifts spaces, *IEEE Trans. Inform. Th.* **57**, no. 6, 3677-3691, 2011.
5. F. Blanchard and G. Hansel, Systèmes codés, *Comp. Sci.* **44** (1986), 17-49.
6. V. Climenhaga and D.J. Thompson, Intrinsic ergodicity beyond specification: β-Shifts, S-gap shifts, their factors, To appear in Israel Journal of Mathematics.
7. J. Franks, Flow equivalence of subshifts of finite type, *Ergo. Th. & Dynam. Sys.* **4** (1984), 53-66.
8. D. Fiebig and U. Fiebig, Covers for coded systems, *Contemporary Mathematics*, **135**, 1992, 139-179.
9. D. Huang, Automorphisms of Bowen-Franks groups of shifts of finite type, *Ergo. Th. & Dynam. Sys.* **21** (2001), no. 4, 1113-1137.
10. U. Jung, On the existence of open and bi-continuous codes, *Trans. Amer. Math. Soc.* **363** (2011), 1399-1417.
11. D. Lind and B. Marcus, *An introduction to symbolic dynamics and coding*, Cambridge Univ. Press, 1995.
12. T. Meyerovitch, Tail invariant measures of the Dyck-shift and non-sofic systems, M.Sc. Thesis, Tel-Aviv University, 2004.
13. J. Moon and B. Brickner, Maximum transition run codes for data storage systems, *IEEE Trans. Magn.* **32**, no. 5, (1996), 39923994.
14. B. E. Moision and P. H. Siegel, Periodic-finite-type shifts spaces, *Proc. 2001 IEEE Int. Symp. Inform. Theory (ISIT01), Washington DC*, (2001), p. 65.
15. M. Newman, *Integral matrices*, Academic Press, New York, 1972.
16. W. Qun, Y. R. Sheng, Pointwise pseudo-orbit tracing property and chaos, *Journal of Mathematical Research & Exposition*, **28**, No. 2, 413-420 (2008).
17. C. Spandl, Computing the topological entropy of shifts, *Math. Log. Quart.* **53**, no. 4/5, 493-510 (2007).
18. M. Schraudner, One-dimensional projective subdynamics of uniformly mixing $\mathbb{Z}^d$ shifts of finite type, preprint.
19. Klaus Thomsen, On the structure of a sofic shift space, *Trans. Amer. Math. Soc*, **356**, No. 9, (2004) pp. 3557-3619.
20. K. Thomsen, *On the structure of β-shifts*, Algebraic and topological dynamics, Contemp. Math., **385**, *Amer. Math. Soc., Providence, RI*, 2005, pp. 321–332.

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